HYPERSURFACE EXCEPTIONAL SINGULARITIES

SHIHOKO ISHII AND YURI PROKHOROV

Abstract. This paper studies hypersurface exceptional singularities in $\mathbb{C}^n$ defined by non-degenerate function. For each canonical hypersurface singularity, there exists a weighted homogeneous singularity such that the former is exceptional if and only if the latter is exceptional. So we study the weighted homogeneous case and prove that the number of weights of weighted homogeneous exceptional singularities are finite. Then we determine all exceptional singularities of the Brieskorn type of dimension 3.

1. Introduction

The notion of exceptional singularities was introduced and its 2-dimensional examples were given by Shokurov in [20, §5]. At the beginning, there was no non-trivial example of higher dimensional exceptional singularities. But in [14] the first higher dimensional non-trivial examples of exceptional singularities are found and then in [15] 3-dimensional exceptional quotient singularities are characterized. The advantage of distinguishing exceptional and non-exceptional singularities is as follows:

(1) for a non-exceptional singularity, the linear system $|-mK_X|$ is to have a "good" member for some small m (actually, we can take $m \in \{1, 2\}$ in the 2-dimensional case and $m \in \{1, 2, 3, 4, 6\}$ in the 3-dimensional case, see [20, 5.2.3 and Theorem 5.6], [16, Theorem 4.9] and also Proposition 2.3);
(2) exceptional singularities are to be classified.

In this paper, we try to determine hypersurface exceptional singularities defined by non-degenerate function. Since an exceptional singularity is log-canonical, under our situation it must be either strictly log-canonical or canonical. For the trivial case, that is, the strictly log-canonical case, the singularity is exceptional if and only if it is purely elliptic of type $(0, d-1)$, where d is the dimension of the singularity ([6]). (In [6] these terminologies are defined only for an isolated singularity, but these are naturally extended to the case that the log-canonical singular locus is isolated). If $d = 2$, then it is a simple elliptic singularity and there are 3-types $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$ (see [19]). If $d = 3$ then it is a simple $K3$-singularity (c.f. [9]), and isolated weighted homogeneous ones are classified into 95-types in [24]. Here we study the other case in which the singularity is canonical. In the 2-dimensional case, canonical singularities are

hypersurface and there is a plt blow-up which is given by a weighted blow-up. Moreover, the class of exceptional canonical 2-dimensional singularities is bounded: E_6, E_7, E_8. We generalize these facts to the higher dimensional case. A criterion for a canonical singularity to be an exceptional singularity is obtained in [16] by means of plt blow-up. In order to make use of it, we have to construct a plt-blow-up first. In §3, we prove that there exists a weighted blow-up which gives a plt-blow-up. We also prove that for every canonical singularity defined by a non-degenerate function there exists such a singularity defined by a weighted homogeneous polynomial such that the former is exceptional if and only if the latter is exceptional. So we can reduce the problem into the weighted homogeneous case. In §4, we prove the finiteness of the set of all weights of weighted homogeneous hypersurface exceptional singularities of fixed dimension. In §5, we determine all exceptional singularities of the Brieskorn type of dimension 3.

2. Preliminaries

All varieties are defined over $\mathbb{C}$. We use terminologies lc, klt, plt, dlt defined in [11], [12]. We denote a germ of a singularity x in X by (X, x). Let D be a normal subvariety of codimension one on a normal variety X. Assume D is Gorenstein in codimension one. Then by [12, 16.5] there exists a $\mathbb{Q}$-Weil divisor $\mathrm{Diff}_{D/X}(0)$ on D such that:

$$K_D + \mathrm{Diff}_{D/X}(0) = K_X + D|_D.$$

This divisor $\mathrm{Diff}_{D/X}(0)$ is called the *different*. If there is no possibility of confusion, this is written as $\mathrm{Diff}_D(0)$.

Definition 2.1. Let (X, x) be a normal singularity and $D = \sum d_i D_i$ a boundary on X such that $K_X + D$ is log-canonical. The pair (X, D) is said to be *exceptional* if there exists at most one exceptional divisor E over X with discrepancy $a(E, D) = -1$. A log-canonical singularity (X, x) is said to be *exceptional* if every lc pair (X, D) is exceptional.

Definition 2.2. Let (X, x) be a normal singularity and $\varphi\colon Y \to X$ a blow-up such that the exceptional locus of φ contains only one irreducible divisor, say S, and $x \in \varphi(S)$. Then φ is called a *plt blow-up* (resp. *lc blow-up*) of (X, x), if (Y, S) is plt (resp. lc) and $-(K_Y + S)$ is φ-ample.

Proposition 2.3 ([16, Theorem 4.9])**.** *Let (X, x) be a klt singularity and let $\varphi\colon (Y, S) \to X$ be a plt blow-up of (X, x). Then the following are equivalent:*

- (i) *(X, x) is non-exceptional;*
- (ii) *there is a boundary $\Xi \geq \mathrm{Diff}_S(0)$ such that $N(K_S + \Xi)$ is nef (in particular $\mathbb{Q}$-Cartier) and (S, Ξ) is not klt;*

(iii) *(in dimension* 3 *only) there is a regular (i.e.* 1, 2, 3, 4 *or* 6*) complement of* $K_S + \mathrm{Diff}_S(0)$ *which is not klt.*

About the definition of a complement, the reader is asked to refer to 5.13.

Proposition 2.4. *Let* (X, x) *be a log-canonical singularity. If there are two lc blow-ups which are not isomorphic over* X*, then* (X, x) *is not exceptional.*

Proof (cf. [15, 2.7]*).* Assume that (X, x) is exceptional. Let $\varphi\colon (Y, S) \to X$ be a lc blow-up. Since $-(K_Y + S)$ is φ-ample, the linear system $-n(K_Y + S)$ is base point free over X for $n \gg 0$. Let $H \in |-n(K_Y + S)|$ be a general member and let $B := \frac{1}{n}\varphi(H)$. By Bertini Theorem, $K_Y + S + \frac{1}{n}H$ is lc. Since $K_Y + S + \frac{1}{n}H$ is $\mathbb{Q}$-linearly trivial, $K_X + B$ is lc and $a(S^{(i)}, B) = -1$ for any component $S^{(i)}$ of S. Hence S is irreducible and $a(S, B) = -1$. If $\varphi'\colon (Y', S') \to X$ is another lc blow-up, then similarly S' is irreducible and we have a boundary B' such that $K_X + B'$ is lc and $a(S', B') = -1$. We claim that S and S' define different discrete valuations of the function field $K(X)$. Indeed, otherwise $\chi\colon Y \dashrightarrow Y'$ is an isomorphism in codimension one and $\chi_*(K_Y + S) = K_{Y'} + S'$. Since both $-(K_Y + S)$ and $-(K_{Y'} + S')$ are ample over X, we have $Y \simeq Y'$. We may assume also that B and B' have no common components.

For $0 \le \alpha \le 1$, define the linear function $\varsigma(\alpha)$ by

$$a(S, \alpha B + \varsigma(\alpha)B') = a(S, 0) - \alpha\, \mathrm{mult}_S(B) - \varsigma(\alpha)\, \mathrm{mult}_S(B') = -1$$

and put $B(\alpha) := \alpha B + \varsigma(\alpha)B'$. Clearly, $\varsigma(1) = 0$ and $B(1) = B$ (because $a(S, B) = -1$). We claim that $K_X + B(\alpha)$ is lc for all $0 \le \alpha \le 1$. Assume the opposite. Then

$$0 < \alpha_0 := \inf_{\alpha \in [0,1]} \{\alpha \mid K_X + B(\alpha) \quad \text{is lc}\}.$$

Fix some log resolution of $(X, B + B')$ factoring through Y and let $E_1, \dots, E_m$ be the new exceptional divisors. The value α_0 can be computed from a finite number of linear inequalities $a(E_i, B(\alpha)) \ge -1$. Therefore α_0 is rational and $K_X + B(\alpha_0)$ is lc. Since $K_X + B(\alpha_0 - \varepsilon)$ is not lc for any $\varepsilon > 0$, some inequality $a(E_i, B(\alpha)) \ge -1$ is an equality (see [11, 3.12]). Hence $a(E_i, B(\alpha_0)) = a(S, B(\alpha_0)) = -1$. This contradicts the exceptionality of (X, x). Thus $K_X + B(\alpha)$ is lc for all $0 \le \alpha \le 1$. In particular, $K_X + \varsigma(0)B'$ is lc. Since (X, x) is exceptional, $a(S, B') > -1$ and therefore $\varsigma(0) > 1$. On the other hand, $a(S', B') = -1$ and $a(S', \varsigma(0)B') < -1$, a contradiction. □

2.5. We make use of toric geometry, and terminologies in [5] are used here. Let N be a free abelian group $\mathbb{Z}^n$ and M its dual $Hom_{\mathbb{Z}}(N, \mathbb{Z})$. Denote $N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M \otimes_{\mathbb{Z}} \mathbb{R}$ by $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$ respectively. We have a canonical pairing $\langle\ \rangle\colon N_{\mathbb{R}} \times M_{\mathbb{R}} \to \mathbb{R}$. Let σ be the positive quadrant $\mathbb{R}^n_{\ge 0}$ of $N_{\mathbb{R}} = \mathbb{R}^n$ and $\sigma^\vee$ its dual. Then $\mathbb{C}^n$ is the toric variety corresponding to the cone σ. For a fan

Δ in N, the corresponding toric variety is denoted by $T_N(\Delta)$. For a primitive element $\mathbf{p} \in N$ of a 1-dimensional cone $\tau = \mathbb{R}_{\geq 0}\mathbf{p}$ in Δ, the closure $\overline{orb(\mathbb{R}_{\geq 0}\mathbf{p})}$ is denoted by $D_{\mathbf{p}}$, which is a divisor on $T_N(\Delta)$.

Definition 2.6. A monomial $x_1^{m_1} \cdots x_n^{m_n} \in \mathbb{C}[[x_1, x_2, \ldots, x_n]]$ is denoted by x^m, where $m = (m_1, \cdots, m_n) \in \mathbb{Z}^n = M$. For a power series $f = \sum_m a_m x^m \in \mathbb{C}[[x_1, x_2, \ldots, x_n]]$, we write $x^m \in f$, if $a_m \neq 0$. For $\mathbf{p} \in N_{\mathbb{R}}$ and a power series f, we define

$$\mathbf{p}(f) = \min_{x^m \in f} \langle \mathbf{p}, m \rangle.$$

We denote the leading term $\sum_{\langle \mathbf{p}, m \rangle = \mathbf{p}(f)} a_m x^m$ of f with respect to $\mathbf{p}$ by $f_{\mathbf{p}}$.

Definition 2.7. For a power series $f = \sum_m a_m x^m \in \mathbb{C}[[x_1, x_2, \ldots, x_n]]$, define the *Newton polyhedron* $\Gamma_+(f)$ in $M_{\mathbb{R}}$ as follows:

$$\Gamma_+(f) = \text{the convex hull of } \bigcup_{x^m \in f} (m + \sigma^\vee).$$

The set of the interior points of $\Gamma_+(f)$ is denoted by $\Gamma_+(f)^0$. For each face γ of $\Gamma_+(f)$, we define the polynomial f_γ as follows:

$$f_\gamma = \sum_{m \in \gamma} a_m x^m.$$

A power series f is said to be *non-degenerate*, if for every face γ the equation $f_\gamma = 0$ defines a hypersurface smooth in the complement of the hypersurface $x_1 \cdots x_n = 0$.

Proposition 2.8 ([22, 10.3])**.** *Let $(X, 0) \subset (\mathbb{C}^n, 0)$ be a singularity defined by a non-degenerate power series f. Then there exists a subdivision Δ of σ in N such that the toric morphism $\varphi\colon T_N(\Delta) \to \mathbb{C}^n$ gives an embedded resolution of $(\mathbb{C}^n, X)$ such that the exceptional set is of pure codimension one and the union of the proper transform of X and the exceptional divisor is of normal crossings.*

Here we state a well-known criterion for a non-degenerate hypersurface singularity to be canonical.

Proposition 2.9 (see for example, [23] or [13, Theorem 3] or [7, Corollary 1.7])**.** *Let $(X, 0) \subset (\mathbb{C}^n, 0)$ be a normal hypersurface singularity defined by a non-degenerate power series f and $\Gamma_+(f)$ its Newton polyhedron. Then $(X, 0)$ is canonical (resp. log-canonical or lc) if and only if $\mathbf{1} = (1, 1, \ldots, 1) \in \Gamma_+(f)^0$ (resp. $\mathbf{1} = (1, 1, \ldots, 1) \in \Gamma_+(f)$), which is equivalent to that $\mathbf{q}(f) < \langle \mathbf{q}, \mathbf{1} \rangle$ (resp. $\mathbf{q}(f) \leq \langle \mathbf{q}, \mathbf{1} \rangle$) for all $\mathbf{q} \in \mathbb{R}^n_{\geq 0}$.*

3. PLT BLOW-UPS OF HYPERSURFACE CANONICAL SINGULARITIES

3.1. Let $(X,0) \subset (\mathbb{C}^n,0)$ be a hypersurface singularity and $\mathbf{p} = (p_1, \dots p_n)$ a primitive element in N with $p_i > 0$ for all i (such $\mathbf{p}$ is called a *weight*). Let $\varphi \colon \mathbb{C}^n(\mathbf{p}) \to \mathbb{C}^n$ be the weighted blow-up with a weight $\mathbf{p}$. Denote the proper transform of X on $\mathbb{C}^n(\mathbf{p})$ by $X(\mathbf{p})$. The weighted blow-up $\varphi \colon \mathbb{C}^n(\mathbf{p}) \to \mathbb{C}^n$ and its restriction $X(\mathbf{p}) \to X$ are sometimes called the $\mathbf{p}$-blow-up. The $\mathbf{p}$-blow-up $\varphi \colon \mathbb{C}^n(\mathbf{p}) \to \mathbb{C}^n$ is obtained by a subdivision of the cone σ. The corresponding fan consists of the faces of cones σ_i $(i = 1, \dots, n)$, where σ_i is generated by $\mathbf{e}_1, \dots, \mathbf{e}_{i-1}, \mathbf{p}, \mathbf{e}_{i+1}, \dots, \mathbf{e}_n$. Here $\mathbf{e}_1, \dots, \mathbf{e}_n$ are the unit vectors $(1,0,\dots,0), \dots, (0,\dots,0,1)$ which generate σ.

Lemma 3.2. *Under the notation of 3.1, let $(X,0)$ be canonical and defined by a non-degenerate power series f.*

(i) *If $\mathbf{1} \in \Gamma_+(f_\mathbf{p})$, then $(\mathbb{C}^n(\mathbf{p}), X(\mathbf{p}) + D_\mathbf{p})$ is lc and $X(\mathbf{p})$ is normal.*

(ii) *If moreover $\mathbf{1} \in \Gamma_+(f_\mathbf{p})^0$, then $(\mathbb{C}^n(\mathbf{p}), X(\mathbf{p}) + D_\mathbf{p})$ is dlt and $\mathrm{Diff}_{X(\mathbf{p})}(0) = 0$.*

Proof. Let $\psi \colon T_N(\Delta) \to \mathbb{C}^n(\mathbf{p})$ be a toric morphism which is a log-resolution of $(\mathbb{C}^n(\mathbf{p}), X(\mathbf{p}) + D_\mathbf{p})$. Denote

$$K_{T_N(\Delta)} + X' + D_\mathbf{p} = \psi^*(K_{\mathbb{C}^n(\mathbf{p})} + X(\mathbf{p}) + D_\mathbf{p}) + \sum_{\mathbb{R}_{\geq 0}\mathbf{q} \in \Delta(1), \mathbf{q} \neq \mathbf{p}} \alpha_\mathbf{q} D_\mathbf{q},$$

where X' is the proper transform of X on $T_N(\Delta)$ and by abuse of notation the divisors corresponding to $\mathbf{p}$ on $\mathbb{C}^n(\mathbf{p})$ and on $T_N(\Delta)$ are both denoted by $D_\mathbf{p}$. Here we may assume that $\mathbf{q}$ are all primitive. Then if $\mathbf{q} \in \sigma_i$, we obtain that

$$\alpha_\mathbf{q} = \langle \mathbf{q}, \mathbf{1} \rangle - \mathbf{q}(f) - \frac{q_i}{p_i}(\langle \mathbf{p}, \mathbf{1} \rangle - \mathbf{p}(f)) - 1.$$

This is because the discrepancy of $K_{\mathbb{C}^n(\mathbf{p})} + X(\mathbf{p})$ at $D_\mathbf{q}$ is

$$a(D_\mathbf{q}, X(\mathbf{p})) = \langle \mathbf{q}, \mathbf{1} \rangle - \mathbf{q}(f) - 1 - \frac{q_i}{p_i}(\langle \mathbf{p}, \mathbf{1} \rangle - \mathbf{p}(f) - 1)$$

by [7, 2.6] and the coefficient of $D_\mathbf{q}$ in $\psi^* D_\mathbf{p}$ is q_i/p_i. Now consider the discrepancy $\alpha_\mathbf{q}$. Since $\langle \mathbf{q}, \mathbf{1} \rangle - \mathbf{q}(f) > 0$ by Proposition 2.9, it follows that $\alpha_\mathbf{q} \geq 0$, if $q_i = 0$. In case $q_i > 0$, define $\mathbf{a} := (a_1, \dots, a_n)$ from the proportion $(q_1 : \dots : q_n) = (p_1 + a_1 : \dots : p_n + a_n)$. Then $a_i = 0$ and $a_j \geq 0$ for $j \neq i$. Since $\mathbf{q}(f) \leq \mathbf{q}(f_\mathbf{p})$, it follows that

$$\frac{\langle \mathbf{q}, \mathbf{1} \rangle - \mathbf{q}(f)}{q_i} \geq \frac{\langle \mathbf{q}, \mathbf{1} \rangle - \mathbf{q}(f_\mathbf{p})}{q_i} = \frac{\langle \mathbf{p} + \mathbf{a}, \mathbf{1} \rangle - (\mathbf{p} + \mathbf{a})(f_\mathbf{p})}{p_i}.$$

Here if one assumes that $\mathbf{1} \in \Gamma_+(f_\mathbf{p})$, the right hand side of the inequality above is

$$\frac{\langle \mathbf{p}, \mathbf{1} \rangle - \mathbf{p}(f_\mathbf{p})}{p_i} + \frac{\langle \mathbf{a}, \mathbf{1} \rangle - \mathbf{a}(f_\mathbf{p})}{p_i} \geq \frac{\langle \mathbf{p}, \mathbf{1} \rangle - \mathbf{p}(f_\mathbf{p})}{p_i} = \frac{\langle \mathbf{p}, \mathbf{1} \rangle - \mathbf{p}(f)}{p_i},$$

which yields that $\alpha_{\mathbf{q}} \geq -1$. If one assumes that $\mathbf{1} \in \Gamma_+(f_{\mathbf{p}})^0$, then the inequality above is strict, since $\langle \mathbf{a}, \mathbf{1}\rangle - \mathbf{a}(f_{\mathbf{p}}) > 0$. So the first assertions of (i) and (ii) are proved. For the second assertion of (i), consider the discrepancy $a(D_{\mathbf{q}}, X(\mathbf{p}))$. This value is $\alpha_{\mathbf{q}} + q_i/p_i$ which is greater than -1, if $q_i \neq 0$ and non-negative if $q_i = 0$ by the argument above. Hence $(\mathbb{C}^n(\mathbf{p}), X(\mathbf{p}))$ is plt. Then, by the inversion of adjunction ([12, 17.6] or [7, 2.7]), $X(\mathbf{p})$ is normal and $K_{X(\mathbf{p})} + \mathrm{Diff}_{X(\mathbf{p})}(0)$ is klt. For the second assertion of (ii), note that $(\mathbb{C}^n(\mathbf{p}), X(\mathbf{p}) + D_{\mathbf{p}})$ is dlt and $\mathbb{C}^n(\mathbf{p})$ is smooth outside of $D_{\mathbf{p}}$. Then $\mathbb{C}^n(\mathbf{p})$ is smooth at a general point of $X(\mathbf{p}) \cap D_{\mathbf{p}}$ by the classification of 2-dimensional log-canonical pairs ([10, 9.6]). Therefore $\mathbb{C}^n(\mathbf{p})$ is smooth in codimension two along $X(\mathbf{p})$, which yields $\mathrm{Diff}_{X(\mathbf{p})}(0) = 0$. □

Proposition 3.3. *Under the notation of 3.1, let $(X, 0)$ be canonical and defined by a non-degenerate power series f.*

(i) *If $\mathbf{1} \in \Gamma_+(f_{\mathbf{p}})$ and $\mathrm{Diff}_{X(\mathbf{p})}(0) = 0$, then the singularity $(X_{\mathbf{p}}, 0) \subset (\mathbb{C}^n, 0)$ defined by $f_{\mathbf{p}}$ is log-canonical and $X(\mathbf{p}) \to X$ is an lc blow-up.*

(ii) *If $\mathbf{1} \in \Gamma_+(f_{\mathbf{p}})^0$, then the singularity $(X_{\mathbf{p}}, 0) \subset (\mathbb{C}^n, 0)$ is again canonical and $X(\mathbf{p}) \to X$ is a plt blow-up.*

Proof. For the first assertions of (i) and (ii), by Proposition 2.9, it is sufficient to prove that $(X_{\mathbf{p}}, 0)$ is smooth in codimension one under the conditions $\mathrm{Diff}_{X(\mathbf{p})}(0) = 0$ and $\mathbf{1} \in \Gamma_+(f_{\mathbf{p}})$. If $X_{\mathbf{p}}$ has a singular locus in codimension one, then $S = D_{\mathbf{p}} \cap X(\mathbf{p})$ has a singular locus in codimension one. Indeed, the restriction $\pi' \colon X_{\mathbf{p}} \setminus \{0\} \to S$ of the canonical projection $\pi \colon \mathbb{C}^n \setminus \{0\} \to \mathbb{P}(p_1, \dots, p_n) = D_{\mathbf{p}}$ is an open map. Then we obtain that π' is a smooth morphism on the smooth locus of S, as $X_{\mathbf{p}}$ is a Cohen-Macaulay variety and each fiber of π' is smooth. Here the singular locus of S is contained in the invariant divisor $(x_1 \cdots x_n = 0) \subset D_{\mathbf{p}}$, because S is defined by $f_{\mathbf{p}}$ in the weighted projective space $D_{\mathbf{p}} = \mathbb{P}(p_1, \dots, p_n)$ and f is non-degenerate. Let Σ be a 1-codimensional component of the singular locus of S and contained in an invariant divisor $C = \{x_i = 0\}$ of $D_{\mathbf{p}}$. For simplicity, let $i = 1$. Let $\psi \colon T_N(\Delta) \to \mathbb{C}^n(\mathbf{p})$ be a log-resolution of $(\mathbb{C}^n(\mathbf{p}), X(\mathbf{p}) + D_{\mathbf{p}})$ which factors through the blow-up of C and X' the proper transform of $X(\mathbf{p})$. Since $\mathrm{Diff}_{X(\mathbf{p})}(0) = 0$, $(X(\mathbf{p}), S)$ is lc by the previous lemma and the adjunction. From the classification of 2-dimensional log-canonical pairs ([10, 9.6]) there exists an exceptional divisor $D_{\mathbf{q}}$ mapped onto C such that the discrepancy $a(D_{\mathbf{q}}|_{X'}, S) = -1$. If $D_{\mathbf{q}}$ is mapped onto $C = \{x_1 = 0\}$, it follows that $\mathbf{q} = \lambda\mathbf{p} + \mu\mathbf{e}_1$, where λ, μ are positive rational numbers. Since $a(D_{\mathbf{q}}|_{X'}, S) = \alpha_{\mathbf{q}} = \langle \mathbf{q}, \mathbf{1}\rangle - \mathbf{q}(f) - (q_i/p_i)(\langle \mathbf{p}, \mathbf{1}\rangle - \mathbf{p}(f)) - 1 = -1$ for $i \neq 1$, it follows that $\langle \mathbf{e}_1, \mathbf{1}\rangle - \mathbf{e}_1(f_{\mathbf{p}}) = 0$, which shows that the order of $f_{\mathbf{p}}$ is 1, a contradiction. Now the first assertions of (i) and (ii) are proved. In particular $f_{\mathbf{p}}$ is irreducible, therefore S is irreducible and reduced. Noting that

$-(K_{X(\mathbf{p})}+S) = (\mathbf{p}(f)-\langle \mathbf{p}, \mathbf{1}\rangle)D_{\mathbf{p}}|_{X(\mathbf{p})}$ is φ-ample, one obtains that $X(\mathbf{p}) \to X$ is a lc blow-up. If $\mathbf{1} \in \Gamma_+(f_{\mathbf{p}})^0$, then, by Lemma 3.3, (ii), $(X(\mathbf{p}), S)$ is dlt, therefore it is plt.

□

Corollary 3.4. *Let $(X, 0) \subset (\mathbb{C}^n, 0)$ be a canonical singularity defined by a non-degenerate weighted homogeneous polynomial of weight $\mathbf{p}$. Then $\mathbf{p}$-blow-up $X(\mathbf{p}) \to X$ gives a plt blow-up of $(X, 0)$.*

Proof. In this case, $f = f_{\mathbf{p}}$, therefore the condition of (ii) of Proposition 3.3 holds. □

Theorem 3.5. *Let $(X, 0) \subset (\mathbb{C}^n, 0)$ be an arbitrary hypersurface canonical singularity defined by a non-degenerate power series f. Then*

(i) *there exists a weight $\mathbf{p}$ such that the weighted blow-up $X(\mathbf{p}) \to X$ gives a plt blow-up of $(X, 0)$ and*

(ii) *for a weight $\mathbf{p}$ obtained in* (i)*, a singularity $(X_{\mathbf{p}}, 0)$ defined by a weighted homogeneous polynomial $f_{\mathbf{p}}$ is again canonical and $(X_{\mathbf{p}}, 0)$ is exceptional if and only if $(X, 0)$ is exceptional.*

Proof. For the statement (i), it is sufficient to show the existence of a compact face γ of $\Gamma_+(f)$ such that $\mathbf{1} \in \Gamma_+(f_\gamma)^0$, because for every compact face γ there exists a weight $\mathbf{p}$ such that $f_\gamma = f_{\mathbf{p}}$. Since

$$\Gamma_+(f) = \bigcup_{\gamma:\text{compact face of } \Gamma_+(f)} \Gamma_+(f_\gamma),$$

there exists a compact face γ such that $\mathbf{1} \in \Gamma_+(f_\gamma)$. Assume that $\mathbf{1}$ is a boundary point of $\Gamma_+(f_\gamma)$ for every such γ as above. Then $\mathbf{1}$ belongs to the non-compact face of $\Gamma_+(f_\gamma)$. So for any such γ there exists i_γ such that $H_{i_\gamma} := \{(m_1, \dots, m_n) \in M_{\mathbb{R}} \mid m_{i_\gamma} \geq 1, m_j \geq 0 \ (j \neq i_\gamma)\}$ contains $\Gamma_+(f_\gamma)$ and $\mathbf{1}$ is on the boundary of H_{i_γ}. Since $\mathbf{1}$ is on the boundary of $\bigcup_{\mathbf{1}\in\Gamma_+(f_\gamma)} H_{i_\gamma}$, it is on the boundary of $\bigcup_{\mathbf{1}\in\Gamma_+(f_\gamma)} \Gamma_+(f_\gamma)$, therefore on the boundary of $\bigcup_{\gamma:\text{compact face of } \Gamma_+(f)} \Gamma_+(f_\gamma)$, a contradiction. For the first statement of (ii), note that one can take a weight $\mathbf{p}$ such that $\mathbf{1} \in \Gamma_+(f_{\mathbf{p}})^0$ by the argument above and apply Proposition 3.3. For such $\mathbf{p}$, denote the proper transform of $X_{\mathbf{p}}$ under the $\mathbf{p}$-blow-up $\varphi\colon \mathbb{C}^n(\mathbf{p}) \to \mathbb{C}^n$ by $X_{\mathbf{p}}(\mathbf{p})$ and $X_{\mathbf{p}}(\mathbf{p}) \cap D_{\mathbf{p}}$ by $S_{\mathbf{p}}$. Then in $D_{\mathbf{p}}$, S and $S_{\mathbf{p}}$ coincide and $\mathrm{Diff}_{S/X(\mathbf{p})}(0)$ and $\mathrm{Diff}_{S_{\mathbf{p}}/X_{\mathbf{p}}(\mathbf{p})}(0)$ coincide, because the both are equal to $\mathrm{Diff}_{S/D_{\mathbf{p}}}(0) + \mathrm{Diff}_{D_{\mathbf{p}}/\mathbb{C}^n(\mathbf{p})}(0)|_S$. Therefore the conditions for $(X, 0)$ and $(X_{\mathbf{p}}, 0)$ to be non-exceptional are the same (Proposition 2.3). □

On determining exceptional hypersurface singularities, now we can reduce the problem into the weighted homogeneous case.

4. Weights of weighted homogeneous exceptional singularities

Lemma 4.1. *Let $(X,0) \subset (\mathbb{C}^n,0)$ be a hypersurface singularity defined by a power series f and let $\mathbf{p}$, $\mathbf{q}$ be two weights such that neither $f_{\mathbf{p}}$ nor $f_{\mathbf{q}}$ is a power of a single coordinate. If there is an isomorphism $X(\mathbf{p}) \simeq X(\mathbf{q})$ over X, then $\mathbf{p} = \mathbf{q}$.*

Proof. Let $\psi \colon \mathbb{C}^n(\mathbf{p})(\mathbf{q}) \to \mathbb{C}^n(\mathbf{p})$ be the toric morphism corresponding to the star-shaped decomposition by adding a 1-dimensional cone $\mathbb{R}_{\geq 0}\mathbf{q}$ and X' be the proper transform of $X(\mathbf{p})$ by ψ. First we show that $X(\mathbf{p}) \cap \psi(D_{\mathbf{q}}) = \varnothing$. Assume the contrary. If $D_{\mathbf{p}} \cap X(\mathbf{p}) \not\subset \psi(D_{\mathbf{q}})$, then $D_{\mathbf{p}}|_{X'}$ and $D_{\mathbf{q}}|_{X'}$ are $\mathbb{Q}$-divisors with different supports, a contradiction to $X(\mathbf{p}) \simeq X(\mathbf{q})$. If $D_{\mathbf{p}} \cap X(\mathbf{p}) \subset \psi(D_{\mathbf{q}})$, then both sides coincide, because the left hand side is of dimension $n-2$, while the right hand side is irreducible and of dimension $\leq n-2$. Therefore the support of $D_{\mathbf{p}} \cap X(\mathbf{p})$ is an invariant divisor on $D_{\mathbf{p}}$, which yields that $f_{\mathbf{p}}$ is a power of a single coordinate, a contradiction. Now we obtain $X(\mathbf{p}) \cap \psi(D_{\mathbf{q}}) = \varnothing$. It implies that the coefficient of $D_{\mathbf{q}}$ in $\psi^*(X(\mathbf{p}))$ is 0. Let $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{q} = (q_1, \ldots, q_n)$. If we put $q_1/p_1 := \min_i(q_i/p_i)$, then $\mathbf{q} \in \sigma_1$ and the coefficient of $D_{\mathbf{q}}$ in $\psi^*(X(\mathbf{p}))$ is $\mathbf{q}(f) - \mathbf{p}(f)(q_1/p_1)$ by [7, 2.6, (2)]. Hence $\mathbf{q}(f) - \mathbf{p}(f)(q_i/p_i) \leq 0$ for all i. Now we obtain that $\mathbf{q}(f)/q_i \leq \mathbf{p}(f)/p_i$ for all i. By making the same procedure with exchanging the role of $\mathbf{p}$ and $\mathbf{q}$, we obtain the opposite inequality $\mathbf{q}(f)/q_i \geq \mathbf{p}(f)/p_i$ for all i, which yields $\mathbf{p} = \mathbf{q}$. □

For the assertion of the lemma above, the condition of $f_{\mathbf{p}}$ and $f_{\mathbf{q}}$ is necessary. In fact we have the following example.

Example 4.2 (by Tomari)**.** Let $(X,0) \subset (\mathbb{C}^3,0)$ be a singularity defined by $f = x_1^k + x_2^{k+1} + x_3^{k+1} = 0$ for $k \geq 3$. Take $\mathbf{p} = (1,1,1)$, $\mathbf{q} = (k+1,k,k)$. Then $f_{\mathbf{q}} = f$ is irreducible and $f_{\mathbf{p}} = x_1^k$ is a power of a single coordinate x_1. And both $X(\mathbf{p})$, $X(\mathbf{q})$ are isomorphic to the canonical model over X.

Proposition 4.3. *For an exceptional canonical singularity $(X,0) \subset (\mathbb{C}^n,0)$ defined by a non-degenerate power series f, a weight which gives a plt blow-up is unique. Therefore a weight $\mathbf{p}$ such that $\mathbf{1} \in \Gamma_+(f_{\mathbf{p}})^0$ is unique. A fortiori, a set $\{m \mid x^m \in f_{\mathbf{p}}\}$ spans a hyperplane in $M_{\mathbb{R}} \simeq \mathbb{R}^n$.*

Proof. By Proposition 2.4, plt bow-up for $(X,0)$ is unique. If a weight $\mathbf{p}$ gives a plt blow-up, then by (ii) of Theorem 3.5, $f_{\mathbf{p}}$ is not a power of a single coordinate. Then apply Lemma 4.1. □

To prove the finiteness of the number of weights for exceptional singularities defined by weighted homogeneous functions, we need the following lemma:

Lemma 4.4. *Let $(X,0) \subset (\mathbb{C}^n, 0)$ be a canonical singularity defined by a non-degenerate power series f. Let g be an irreducible weighted homogeneous polynomial with weight $\mathbf{p}$. Assume that $\mathbf{1} \in \Gamma_+(f_{\mathbf{p}})$ and $x^m \in g$ for every m with $x^m \in f_{\mathbf{p}}$. Then $\mathbf{p}$-blow-up $X(\mathbf{p}) \to X$ is a lc blow-up.*

Proof. By (i) in Proposition 3.3, it is sufficient to prove that $\mathrm{Diff}_{X(\mathbf{p})}(0) = 0$. Assume that $\mathrm{Diff}_{X(\mathbf{p})}(0) \neq 0$. Then a 2-codimensional irreducible component of the singular locus of $\mathbb{C}^n(\mathbf{p})$ is contained in $X(\mathbf{p})$, which implies that $\{x_i = 0\}$ in $D_{\mathbf{p}}$ is contained in $\{f_{\mathbf{p}} = 0\}$ in $D_{\mathbf{p}}$ for some i. Therefore $x_i \mid f_{\mathbf{p}}$. By $\mathbf{1} \in \Gamma_+(f_{\mathbf{p}})$, it follows that $x_i^2 \nmid f_{\mathbf{p}}$. For the simplicity, let $i = 1$. Here $\mathbb{C}^n(\mathbf{p})$ is singular along $\{x_1 = 0\}$ in $D_{\mathbf{p}}$, so the weight $\mathbf{p}$ is represented by $(p_1, ap'_2, \ldots, ap'_n)$ for some integer $a \geq 2$ with $a \nmid p_1$. Now, by the assumption on g, it follows that $g = x_1 g' + g''$, where $x_1 \nmid g'$ and x_1 does not appear in g'', as g is irreducible. Then $\mathbf{p}(g) = \mathbf{p}(x_1 g') \equiv p_1 \mod a$ and $\mathbf{p}(g) = \mathbf{p}(g'') \equiv 0 \mod a$, which is a contradiction. □

Theorem 4.5. *For a fixed n, the number of weights of non-degenerate weighted homogeneous polynomials which define exceptional canonical singularities in $(\mathbb{C}^n, 0)$ is finite.*

Proof. Let $\mathcal{F} = \{f \mid$ a non-degenerate weighted homogeneous power series defining an exceptional canonical singularity in $\mathbb{C}^n$ at $0\}$ and $\mathbb{W} = \{$ the weight of $f \mid f \in \mathcal{F}\}$. Assume that the set $\mathbb{W}$ is infinite and induce a contradiction. For $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n_{\geq 0}$, define $|m| := \sum_{i=1}^n m_i$.

Step 1. For $f \in \mathcal{F}$, define $\alpha_f := \min\{|m| \mid x^m \in f\}$. Then $\{\alpha_f\}_{f \in \mathcal{F}}$ is bounded by n from above, since $\mathbf{1} \in \Gamma_+(f)^0$. Hence there exist a subset $\mathcal{F}_1 \subset \mathcal{F}$ and $m^{(1)} \in \mathbb{Z}^n_{\geq 0}$ such that $\mathbb{W}_1 := \{$ the weight of $f \mid f \in \mathcal{F}_1\}$ is a infinite set and $x^{m^{(1)}} \in f$, $|m^{(1)}| = \alpha_f$ for any $f \in \mathcal{F}_1$, because $\{m \in \mathbb{Z}^n_{\geq 0} \mid |m| \leq n\}$ is finite and $\mathbb{W}$ is infinite. Therefore every $f \in \mathcal{F}_1$ is written as $f = a_1 x^{m^{(1)}} + g$, $(x^{m^{(1)}} \notin g,\ a_1 \in \mathbb{C})$.

Step2. For $f = a_1 x^{m^{(1)}} + g \in \mathcal{F}_1$, define $\beta_f := \min\{|m| \mid x^m \in g\}$. If $\{\beta_f\}_{f \in \mathcal{F}_1}$ is bounded, then, by taking infinite sets $\mathcal{F}_2$ and $\mathbb{W}_2$ smaller, we can write $f = a_1 x^{m^{(1)}} + a_2 x^{m^{(2)}} + g_2$ for every $f \in \mathcal{F}_2$ in the same way as in Step 1. In particular, if $x^{m^{(1)}}$ is a power of a single coordinate, then $\{\beta_f\}$ is bounded. Indeed if $\{\beta_f\}$ is unbounded, then $\bigcap_{f \in \mathcal{F}_1} \Gamma_+(f) = \Gamma_+(x^{m^{(1)}})$, where $\mathbf{1}$ belongs to the left hand side and not to the right hand side, a contradiction.

Step 3. By the successive procedures, one obtains infinite sets $\mathcal{F}_r$, $\mathbb{W}_r$ such that for every $f \in \mathcal{F}_r$, $f = a_1 x^{m^{(1)}} + \ldots + a_r x^{m^{(r)}} + h$ and $\{\gamma_f\}_{f \in \mathcal{F}_r}$ is unbounded, where $\gamma_f := \min\{|m| \mid x^m \in h\}$. Indeed, if these procedures do not terminate, one obtains an infinite series $\{m^{(1)}, m^{(2)}, \ldots, m^{(i)}, \ldots\}$ in $\mathbb{Z}^n_{\geq 0} \subset M_{\mathbb{R}}$. Let L_i be the linear subvariety spanned by $\{m^{(1)}, m^{(2)}, \ldots, m^{(i)}\}$. Then $L_i \neq M_{\mathbb{R}}$ for all i, because $x^{m^{(1)}}, x^{m^{(2)}}, \ldots, x^{m^{(i)}}$ appear in the weighted

homogeneous polynomials in $\mathcal{F}_i$. Since $\ldots \subset L_i \subset L_{i+1} \subset \ldots$, there exists r such that $L_r = L_{r+i}$ for all $i > 0$. Let $\mathbf{p} = (p_1, \ldots, p_r)$ be the weight of an element $f \in \mathcal{F}_r$. Then L_r is contained in a hyperplane $H_{\mathbf{p}} := \{(m_1, \ldots, m_n) \mid \sum p_i m_i = p\}$ for some $p \in \mathbb{N}$. Hence the infinite set $\{m^{(1)}, m^{(2)}, \ldots, m^{(i)}, \ldots\}$ is contained in $H_{\mathbf{p}} \cap \mathbb{Z}^n_{\geq 0}$ which is a finite set because of $p_i > 0$, (i=1, ... ,n), a contradiction.

Step 4. Now fix an element $f' \in \mathcal{F}_r$ and let $\mathbf{p}$ be a weight of f'. As $\{\gamma_f\}_{f \in \mathcal{F}_r}$ is unbounded, one can take $f = a_1 x^{m^{(1)}} + \ldots + a_r x^{m^{(r)}} + h \in \mathcal{F}_r$ such that $\mathbf{p}(h) > \mathbf{p}(f') = \mathbf{p}(x^{m^{(i)}})$ for $i = 1, \ldots, r$. Then $f_{\mathbf{p}} = a_1 x^{m^{(1)}} + \ldots + a_r x^{m^{(r)}}$. So if $x^m \in f_{\mathbf{p}}$, then $x^m \in f'$. On the other hand, from the unboundedness of γ_f, it follows that $\mathbf{1} \in \bigcap_{f'' \in \mathcal{F}_r} \Gamma_+(f'') = \Gamma_+(f_{\mathbf{p}})$. Now by Lemma 4.4, $\mathbf{p}$-blow-up of $X = \{f = 0\}$ is an lc blow-up. On the other hand, for the weight $\mathbf{q}$ of f, $\mathbf{q}$-blow-up of X is a plt blow-up by Corollary 3.4. Since neither $f_{\mathbf{p}}$ nor $f_{\mathbf{q}}$ is a power of a single coordinate, by Lemma 4.1 $X(\mathbf{p}) \not\simeq X(\mathbf{q})$, which is a contradiction to that $(X, 0)$ is exceptional. □

Remark 4.6. The theorem is the same as the finiteness of Newton polyhedrons of such singularities.

Corollary 4.7. *Fix $n \in \mathbb{N}$. Let $(X, 0) \subset (\mathbb{C}^n, 0)$ be a canonical exceptional singularity defined by a non-degenerate power series f and let $\varphi\colon (Y, S) \to X$ be a plt blow-up. Then the pair $(S, \operatorname{Diff}_S(0))$ contained in a finite number of algebraic families.*

Note that this fact is known to be true also for any klt singularity of dimension ≤ 3 (see [21, §4]).

Proof. By Theorems 3.5 and 4.5 we may assume (up to finite numbers of cases) that φ is a weighted blow-up of fixed weight $\mathbf{p}$. Then the exceptional divisor S is defined in the weighted projective space $\mathbb{P}(\mathbf{p})$ by $f_{\mathbf{p}} = 0$. Thus we may assume that S is contained in some algebraic family. Now let H be a very ample divisor on S. Write $\operatorname{Diff}_S(0) = \sum (1 - 1/m_i)\Delta_i$, where $m_i \in \mathbb{N}$ and Δ_i's are prime divisors. Since $-(K_S + \operatorname{Diff}_S(0))$ is ample, Const $\geq H^{n-3} \cdot (-K_S) > \sum (1 - 1/m_i) H^{n-3} \cdot \Delta_i \geq \frac{1}{2} \sum H^{n-3} \cdot \Delta_i$. Thus the degree of components of $\operatorname{Diff}_S(0)$ under the embedding $S \hookrightarrow \mathbb{P}^N$ given by $|H|$ is bounded. Then $\operatorname{Supp}(\operatorname{Diff}_S(0))$ belongs to a finite number of families (see, e.g., [2, Ch. 3 §7]) and we may assume that $(S, \operatorname{Supp}(\operatorname{Diff}_S(0)))$ is fixed. Now we need to show only that $m_i \leq$ Const for all i. Indeed, in the opposite case we can take an infinite sequence $\operatorname{Diff}_S^{(1)}(0) < \operatorname{Diff}_S^{(2)}(0) < \cdots$. Let $\operatorname{Diff}_S^{(\infty)}(0) := \lim \operatorname{Diff}_S^{(k)}(0)$. Then $-(K_S + \operatorname{Diff}_S^{(\infty)}(0))$ is nef and $\left\lfloor \operatorname{Diff}_S^{(\infty)}(0) \right\rfloor \neq 0$. This contradicts Proposition 2.3. □

5. Exceptional canonical singularities of Brieskorn type

The aim of this section is to prove the following:

Theorem 5.1. *Let $X \subset \mathbb{C}^4$ be a hypersurface canonical singularity given by the equation*

$$x_1^{a_1} + x_2^{a_2} + x_3^{a_3} + x_4^{a_4} = 0, \quad a_1 \le a_2 \le a_3 \le a_4. \tag{5.I}$$

Then $(X, 0)$ is exceptional if and only if $[a_1, a_2, a_3, a_4]$ is one of the following:

$[3, 3, 4, d]$,	$5 \le d \le 11$	$[3, 3, 5, d]$,	$d = 6, 7$
$[3, 4, 4, d]$,	$d = 4, 5$	$[2, 3, 7, d]$,	$8 \le d \le 41$
$[2, 3, 8, d]$,	$8 \le d \le 23$	$[2, 3, 9, d]$,	$9 \le d \le 17$
$[2, 3, 10, d]$,	$10 \le d \le 14$	$[2, 3, 11, d]$,	$d = 11, 12, 13$
$[2, 4, 5, d]$,	$6 \le d \le 19$	$[2, 4, 6, d]$,	$6 \le d \le 11$
$[2, 4, 7, d]$,	$d = 7, 8, 9$	$[2, 5, 5, d]$,	$5 \le d \le 9$
$[2, 5, 6, d]$,	$d = 6, 7.$		

Moreover, exceptional divisors $(S, \mathrm{Diff}_S(0))$ of corresponding plt blow-ups are described in Tables 1 and 2.

From Proposition 2.9 (see also [18], [11, 8.14]) we have

Corollary 5.2. *Let $X \subset \mathbb{C}^n$ be a hypersurface singularity*

$$x_1^{a_1} + x_2^{a_2} + \cdots + x_n^{a_n} = 0. \tag{5.II}$$

Then it is canonical (resp. log-canonical) if and only if

$$1/a_1 + 1/a_2 + \cdots + 1/a_n > 1, \qquad \textit{resp.} \qquad \ge 1.$$

Moreover, $K_{\mathbb{C}^n} + X$ is lc if and only if $1/a_1 + \cdots + 1/a_n \ge 1$.

Lemma 5.3. *Let $X \subset \mathbb{C}^n$ be a canonical hypersurface singularity* (5.II), *let $W \subset \mathbb{C}^n$ be the hyperplane $\{x_n = 0\}$ and let $F := W \cap X$. If $1/a_1 + \cdots + 1/a_{n-1} \ge 1$, then $K_X + F$ is lc.*

Proof. By Corollary 5.2, $K_W + F$ is lc. Note that $K_{\mathbb{C}^n} + X$ is plt (by Inversion of Adjunction). Applying [12, 17.7] twice we obtain that both $K_{\mathbb{C}^n} + X + W$ and $K_X + F$ are lc. □

5.4. **Notation.** From now on we assume that $X \subset \mathbb{C}^4$ is a hypersurface canonical singularity given by the equation (5.I). The sequence $[a_1, a_2, a_3, a_4]$ is called the *type* of X. Set $w := \mathrm{lcm}(a_1, \ldots, a_4)$ and consider the weighted blow-up $f \colon Y = X(\mathbf{p}) \to X$, where $p_i := w/a_i$. Since f is weighted homogeneous with respect to $\mathbf{p}$, $f \colon Y \to X$ is a plt blow-up (see Corollary 3.4). Let S be the exceptional divisor. Put $\varDelta := \mathrm{Diff}_S(0)$. Note that S is given in the weighted projective space $\mathbb{P}(\mathbf{p})$ by the equation (5.I). Let $\varGamma_i$, $i = 1, \ldots, 4$ be a curve on $S \subset \mathbb{P}(\mathbf{p})$ which is cut out by $x_i = 0$. Set $\varGamma := \varGamma_1 + \cdots + \varGamma_4$.

Lemma 5.5 (see e.g. [11, 8.16]). *For any $n \in \mathbb{N}$ there is a constant $\delta(n)$ such that for any $a_1, \dots, a_n \in \mathbb{N}$ only one of the following inequalities holds:*

$$\sum_{i=1}^{n} \frac{1}{a_i} \le 1 - \delta(n) \qquad \text{or} \qquad \sum_{i=1}^{n} \frac{1}{a_i} \ge 1.$$

Moreover, $\delta(1) = 1/2$, $\delta(2) = 1/6$, $\delta(3) = 1/42$.

5.5.1. If $(X, 0)$ is exceptional, then by [14, Lemma 1.7], $K_X + W$ is not lc for any Cartier divisor W with $W \ni 0$. Thus 5.2 and Lemma 5.3 give us $1/a_1 + \dots + 1/a_4 > 1$ and $1/a_1 + 1/a_2 + 1/a_3 < 1$. Since $(X, 0)$ is canonical, $a_1 = 2$ or 3 (see 2.9). Then $1/a_2 + 1/a_3 + 1/a_4 > 1/2$ and $a_2 \le 5$. Further, $a_2 \ge 3$ and $1/a_3 + 1/a_4 > 1/6$. Hence, $a_3 \le 11$. Finally, by Lemma 5.5 we have $1/a_1 + 1/a_2 + 1/a_3 \le 41/42$ and $a_4 \le 41$. This yields for $[a_1, a_2, a_3, a_4]$ cases as in Theorem 5.1 and additionally the following cases: $[2, 3, 7, 7]$, $[3, 3, 4, 4]$, $[2, 4, 5, 5]$. In these cases the singularity is not exceptional. This will be proved in 5.12.1, 5.18.1 and 5.18.2, respectively.

Remark. As above, Lemma 5.5 gives an effective bound of canonical exceptional singularities of the Brieskorn type in any dimension.

Lemma 5.6. *Let $S \subset \mathbb{P} = \mathbb{P}(p_1, \dots, p_n)$ be a hypersurface*

$$x_1^{a_1} + \dots + x_n^{a_n} = 0, \quad \text{where} \quad a_1 p_1 = \dots = a_n p_n = w.$$

Let $\mathit{\Gamma}_i := S \cap \{x_i = 0\}$, $i = 1, \dots, n$ and $\mathit{\Gamma} = \sum \mathit{\Gamma}_i$. Then

(i) $\mathrm{Diff}_{S/\mathbb{P}}(0) = 0$;
(ii) *$K_S + \mathit{\Gamma}_i$ is plt for all $i = 1, \dots, n$;*
(iii) *$K_S + \mathit{\Gamma}$ is lc.*

Proof. (i) is obvious. Indeed, codimension two singularities of $\mathbb{P}$ are contained in $\cup_{i \ne j}\{x_i = x_j = 0\}$ and S does not contain its components. To prove (ii) and (iii) we consider the finite map $\mathbb{P}^{n-1} \to \mathbb{P}$ given by

$$(x_1, \dots, x_n) \to (x_1^{p_1}, \dots, x_n^{p_n}).$$

The ramification divisor is $\sum (p_i - 1) H_i$, where H_i is i-th coordinate hyperplane on $\mathbb{P}^{n-1}$. Let $S' = \{x_1^w + \dots + x_n^w = 0\} \subset \mathbb{P}^{n-1}$ be the preimage of S. The restriction $\varphi\colon S' \to S$ is also a finite morphism of degree w. Put $L_i := H_i \cap S'$. By the ramification formula we have

$$\varphi^* \left(K_S + \sum \mathit{\Gamma}_i \right) = K_{S'} + \sum L_i.$$

Since $S' + \sum L_i$ is a normal crossing divisor, $K_{S'} + \sum L_i$ is lc. Then $K_S + \sum \mathit{\Gamma}_i$ is lc by [20, §2] or [12, 20.3]. (ii) can be proved in a similar way. □

Corollary 5.6.1. *Notation as in 5.4. Let Θ be a boundary on S. Assume that* $\mathrm{Supp}(\Theta) \subset \mathit{\Gamma}$ *and $\lfloor \Theta \rfloor = 0$. Then $K_S + \Theta$ is klt.*

Lemma 5.7. *Notation as in 5.4. Then* $\Delta = \sum(1-1/m_i)\Gamma_i$, *where*
$$m_i = \gcd(p_1, \dots, \hat{p}_i, \dots, p_4).$$

Proof. Restricting $K_{\mathbb{C}^4(\mathbf{p})}$ to S we obtain
$$\mathrm{Diff}_{S/Y}(0) + \mathrm{Diff}_{Y/\mathbb{C}^4(\mathbf{p})}(0)|_S = \mathrm{Diff}_{S/\mathbb{P}}(0) + \mathrm{Diff}_{\mathbb{P}/\mathbb{C}^4(\mathbf{p})}(0)|_S.$$
By Lemma 3.2 and Lemma 5.6, $\mathrm{Diff}_{Y/\mathbb{C}^4(\mathbf{p})}(0) = 0$ and $\mathrm{Diff}_{S/\mathbb{P}}(0) = 0$. Hence, $\mathrm{Diff}_{S/Y}(0) = \mathrm{Diff}_{\mathbb{P}/\mathbb{C}^4(\mathbf{p})}(0)|_S$. Taking into account that $\mathrm{Diff}_{\mathbb{P}/\mathbb{C}^4(\mathbf{p})}(0) = \sum(1-1/m_i)\{x_i = 0\}$, we get the assertion. □

We say that expression $\mathbf{p} = (p_1, \dots, p_n)$ is *normalized* if for each i we have $\gcd(p_1, \dots, \hat{p}_i, \dots, p_n) = 1$.

Lemma 5.8 (see, e.g., [4, 1.3.1])**.** (i) *If* $q = \gcd(p_1, \dots, p_n)$, *then* $\mathbb{P}(p_1, \dots, p_n) \simeq \mathbb{P}(p_1/q, \dots, p_n/q)$.

(ii) *If* $\gcd(p_1, \dots, p_n) = 1$ *and* $q = \gcd(p_2, \dots, p_n)$, *then the map*
$$\mathbb{C}^{n+1} \longleftarrow \mathbb{C}^{n+1}, \qquad (x_1^q, x_2, \dots, x_n) \longleftarrow (\overline{x}_1, \overline{x}_2, \dots, \overline{x}_n).$$
induces the isomorphism
$$\mathbb{P}(p_1, p_2, \dots, p_n) \simeq \mathbb{P}(p_1, p_2/q, \dots, p_n/q).$$

Lemma 5.9. *Let* $\mathbb{P} = \mathbb{P}(\mathbf{p})$ *be a weighted projective space, where* $\mathbf{p} = (p_1, \dots, p_n)$. *Assume that* $\mathbf{p}$ *is normalized. Then*

(i) $K_{\mathbb{P}} \sim \mathcal{O}_{\mathbb{P}}(-\sum p_i)$;

(ii) $\mathcal{O}_{\mathbb{P}}(1)^{n-1} = 1/p_1 \cdots p_n$.

Proof. (i) follows, for example, from the discussion 2.1 and 2.2 of [4]. (ii) follows easily from the fact that $\mathbb{P}$ is a finite abelian quotient of the projective space. □

5.10. By Lemma 5.8 we may assume that the exceptional divisor S is given by the equation $y_1^{\overline{a}_1} + \dots + y_4^{\overline{a}_4} = 0$ in $\mathbb{P}(\overline{\mathbf{p}})$, where $\overline{\mathbf{p}} = (\overline{p}_1, \dots, \overline{p}_4)$ is normalized. The algorithm of computation of $\mathbf{p}$ and $(\overline{a}_1, \dots, \overline{a}_4)$ is as follows.

Starting with $[a_1, a_2, a_3, a_4]$ we find $w = \mathrm{lcm}(a_1, \dots, a_4)$, $p_i = w/a_i$. Then $\gcd(p_1, \dots, p_4) = 1$. For convenience we put $a_1, \dots, a_4, p_1, \dots, p_4$ into a (2×4)-matrix and perform the following transformations:

1) $$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ p_1' & p_2' & p_3' & p_4' \end{pmatrix} \longrightarrow \begin{pmatrix} a_1/d' & a_2 & a_3 & a_4 \\ p_1' & p_2'/d' & p_3'/d' & p_4'/d' \end{pmatrix},$$
where $d' := \gcd(p_2', p_3', p_4')$, 2) …, 4) … In four steps we get the matrix
$$\begin{pmatrix} \overline{a}_1 & \overline{a}_2 & \overline{a}_3 & \overline{a}_4 \\ \overline{p}_1 & \overline{p}_2 & \overline{p}_3 & \overline{p}_4 \end{pmatrix}$$
with the normalized second row. Then
$$S = \{y_1^{\overline{a}_1} + \dots + y_4^{\overline{a}_4} = 0\} \subset \mathbb{P}(\overline{\mathbf{p}}).$$

By Lemma 5.7, $\Delta = \sum(1-1/m_i)\Gamma_i$, where $m_i = \gcd(p_1, \dots, \hat{p}_i, \dots, p_4)$.

Note that if $\overline{a}_k = 1$ for some k, then the projection

$$S \longrightarrow \mathbb{P}(\overline{p}_1, \dots, \hat{\overline{p}}_k, \dots, \overline{p}_4) \tag{5.III}$$

is an isomorphism. It is easy to see that this holds if and only if

$$(\star) \qquad \exists k \qquad (a_k, a_i) = 1 \qquad \text{for all} \qquad i \neq k.$$

The following lemma can be easily proved by direct local computations.

Lemma 5.11 (cf. [3])**.** *Let $S \subset \mathbb{P}(\mathbf{p})$ be a hypersurface* (5.I)*, where $\mathbf{p} = (p_1, p_2, p_3, p_4)$, $p_1a_1 = \dots = p_4a_4 = w$ and $\mathbf{p}$ is a weight. Then $\operatorname{Sing}(S) = S \cap \operatorname{Sing}(\mathbb{P}) \subset \cup_{i\neq j}\Gamma_i \cap \Gamma_j$. If $\mathbf{p}$ is normalized, then at points $\Gamma_i \cap \Gamma_j$, $i \neq j$ the surface S has a singularity of type $\frac{1}{d}(p_i, p_j)$, where $d = \gcd(p_k, p_l)$, $\{k, l\} \cap \{i, j\} = \varnothing$. Moreover, $\#\Gamma_i \cap \Gamma_j = w/\operatorname{lcm}(p_k, p_l)$.*

5.12. **Singularities which satisfy** $(\star)$**.** First consider singularities which satisfy the condition $(\star)$. Then $S \simeq \mathbb{P}(\mathbf{q})$, where $\mathbf{q} = (q_1, q_2, q_3) = (p_1, \dots, \hat{p}_k, \dots, p_4)$. Let L_1, L_2, L_3 be the coordinate lines in $\mathbb{P}(\mathbf{q})$. For $m \in \mathbb{N}$ let C_m be a curve in $\mathbb{P}$ given by the equation $x^{m/q_1} + y^{m/q_2} + z^{m/q_3} = 0$ (we assume that $q_i \mid m$ for $i = 1, 2, 3$). Note that C_m is a smooth curve contained in the smooth locus of $\mathbb{P}(\mathbf{q})$. The projection (5.III) identifies $\Gamma_1, \dots, \hat{\Gamma}_k, \dots, \Gamma_4$ with L_1, L_2, L_3 and Γ_k with $C_{\overline{w}}$, where $\overline{w} =: \overline{a}_1\overline{p}_1 = \dots = \overline{a}_4\overline{p}_4$. By H denote the positive generator of the Weil divisor class group of S. If $\mathbf{q}$ is normalized, then $\mathcal{O}(H) = \mathcal{O}(1)$ (see [4]). Recall also that $H^2 = 1/q_1q_2q_3$. Taking into account 5.10 and 5.5.1 we obtain Table 1 and additionally case $[2, 3, 7, 7]$ below.

5.12.1. *Case* $[2, 3, 7, 7]$. Then $S = \mathbb{P}(7, 1, 1)$ and $\Delta = \frac{1}{2}C_7 + \frac{2}{3}L_1$. Take $\Delta^+ = \frac{1}{2}C_7 + \frac{2}{3}L_1 + \frac{5}{6}M$, where $M := \{y + z = 0\}$. Then $K_S + \Delta^+ \sim_{\mathbb{Q}} 0$. It is easy to see that $K_S + \Delta^+$ is not klt at $C_7 \cap L_1 \cap M$. Here the singularity is non-exceptional.

. Now we prove that all singularities in Table 1 are exceptional. We consider them case by case according to the type of the surface S. We will assume that there exists a regular n-complement $K_S + \Delta^+$ and derive a contradiction or prove that $K_S + \Delta^+$ is klt (see 2.3). Set $\Delta' := \Delta^+ - \Delta$. We need the definition and a few properties of complements.

Definition 5.13 ([20])**.** Let S be a normal variety and let $D = C + B$ be a boundary on S, where $C := \lfloor D \rfloor$ and $B := \{D\}$. Then we say that $K_S + D$ is *n-complementary*, if there is a $\mathbb{Q}$-divisor D^+ such that

(i) $n(K_S + D^+) \sim 0$ (in particular, nD^+ is an integral divisor);
(ii) $K_S + D^+$ is lc;
(iii) $nD^+ \geq nC + \lfloor (n+1)B \rfloor$.

In this situation an *n-complement* of $K_S + D$ is $K_S + D^+$. We say that an n-complement is *regular* if $n \in \{1, 2, 3, 4, 6\}$.

Lemma 5.14 ([21], [17]). *Let $K+\sum \delta_i^+ \Delta_i$ be an n-complement of $K+\sum \delta_i \Delta_i$, where Δ_i's are irreducible components.*

(i) *If $\delta_i = 1 - 1/m_i$, $m_i \in \mathbb{N}$, then $\delta_i^+ \geq \delta_i$.*
(ii) *If $\delta_i \geq 6/7$ and $n \in \{1,2,3,4,6\}$, then $\delta_i^+ = 1$.*
(iii) *If $\delta_i = 4/5$ and $n \in \{1,2,3,4,6\}$, then $\delta_i^+ = 5/6$ or 1.*

Cases when $S \simeq \mathbb{P}^2$. For example, assume that $a_2 = a_3 = a_4$. Here $\gcd(a_1, a_2) = 1$. Algorithm 5.10 is as follows:

$$\begin{pmatrix} a_1 & a_2 & a_2 & a_2 \\ a_2 & a_1 & a_1 & a_1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & a_2 & a_2 & a_2 \\ a_2 & 1 & 1 & 1 \end{pmatrix}.$$

Thus $S \simeq \mathbb{P}^2$ and $\Delta = \frac{a_1-1}{a_1} C_{a_2}$. There are two possibilities: $[3,4,4,4]$ and $[2,5,5,5]$. In case $[3,4,4,4]$, among regular complements of $K_S + \Delta$ there are only 3-complement $K_{\mathbb{P}^2} + \frac{2}{3}C_4 + \frac{1}{3}M_1$, where M_1 is a line, 6-complement $K_{\mathbb{P}^2} + \frac{2}{3}C_4 + \frac{1}{6}M_2$, where M_2 is a conic and 6-complement $K_{\mathbb{P}^2} + \frac{2}{3}C_4 + \frac{1}{6}M_1' + \frac{1}{6}M_1''$, where M_1', M_1'' are lines. All these complements are klt by Lemma 5.15 below. Case $[2,5,5,5]$ is similar.

Lemma 5.15 (cf. [16, Lemma 3]). *Let (S, o) be a normal analytic surface germ and let $M = \sum d_i M_i$ be a $\mathbb{Q}$-divisor on S. Assume that $K_S + M_i$ is plt at o for all i (for instance, this holds if (S, o) is smooth and all M_i's also are smooth at o). If $\sum d_i \leq 1$ and $\lfloor M \rfloor \leq 0$, then $K_S + M$ is klt at o.*

Proof. Let $\varphi\colon (S', o') \to (S, o)$ be a finite étale in codimension one cover such that S' is smooth. Set $M' := \varphi^* M$ and $M_i' := \varphi^* M_i$. By [12, 20.4], $K_{S'} + M_i'$ is plt for all i. Hence, all M_i''s are smooth irreducible curves. Again, by [12, 20.4] it is sufficient to show that $K_{S'} + M'$ is klt. By our assumption, $M' = \sum d_i' M_i'$, where $\sum d_i' \leq 1$ and $\lfloor M' \rfloor \leq 0$. Let $\sigma\colon \overline{S} \to S'$ be the blow-up of o' and let $\overline{M}$ be the crepant pull-back of M' (i.e. $\sigma^*(K_{S'} + M') = K_{\overline{S}} + \overline{M}$). By [11, 3.10], $K_{S'} + M'$ is klt if and only if so is $K_{\overline{S}} + \overline{M}$. Clearly, all the irreducible components $\overline{M}_i$ of $\overline{M}$ are smooth. Write $\overline{M} = \sum \overline{d}_i \overline{M}_i$ so that $\overline{M}_0$ is the exceptional divisor and $\overline{M}_i$'s are proper transforms of M_i''s for $i \neq 0$. It is easy to see that $\overline{d}_i = d_i'$ for $i \neq 0$ and $\overline{d}_0 = \sum d_i' - 1 \leq 0$. So we again have $\sum \overline{d}_i \leq 1$. Thus, it is sufficient to prove our statement on $\overline{S}$. We replace S' with $\overline{S}$ and continue the process. At the end, we get the situation when $\operatorname{Supp}(M')$ is a normal crossing divisor. In this situation, the inequalities $\sum d_i' \leq 1$ and $d_i' < 1$ gives us that $K_{S'} + M'$ is klt (and even canonical). □

If $(a_i, a_j) = 1$ for all pairs (i, j), $i \neq j$, then $S \simeq \mathbb{P}^2$ and $\Delta = \sum_{i=1}^3 (1 - 1/a_i) L_i + (1 - 1/a_4) C_1$. For $[a_1, a_2, a_3, a_4]$ there are the following possibilities:

$$[2,3,11,13] \quad \text{and} \quad [2,3,7,r], r \in \{11,13,17,19,23,25,29,31,37,41\}.$$

All these singularities are exceptional. Indeed, if $K_S + \Delta^+$ is a regular complement of $K_S + \Delta$, then $\Delta^+ \geq \Delta$ (see 5.14). Therefore $\Delta^+ \geq \frac{1}{2}L_1 + \frac{2}{3}L_3 + L_3 + C_1$.

But $\deg \varDelta^+ = 3$, a contradiction. Similarly, one can treat the other cases with $S \simeq \mathbb{P}^2$ (see Table 1).

Remark. For any canonical singularity $(X, 0)$, define

$$\mathrm{compl}(X) := \min\{m \mid \text{ there is a non-klt } m\text{-complement of } K_X\}.$$

Let $f\colon (Y, S) \to X$ be a plt blow-up. Then

$$\mathrm{compl}(X) \le \min\{m \mid K_S + \mathrm{Diff}_S(0) \text{ is } m\text{-complementary}\}$$

(see [16, Corollary 1]). Moreover, if $(X, 0)$ is exceptional, then equality holds. It follows from [21] that $\mathrm{compl}(X)$ is bounded in the three-dimensional case. If X is of the Brieskorn type $[2, 3, 11, 13]$, then $\mathrm{compl}(X) = 66$ (see [17]). This is the maximal known value of $\mathrm{compl}(X)$ for three-dimensional canonical singularities. Note that in the two-dimensional case $\mathrm{compl}(X) \le 6$ and the equality achieves for singularities of type E_8 (= Brieskorn type $[2, 3, 5]$). By 2.3, $\mathrm{compl}(X) \in \{1, 2, 3, 4, 6\}$ for any three-dimensional non-exceptional singularity.

Conjecture. *Let (X, o) be a canonical singularity. Then* $\mathrm{compl}(X) \le 66$.

It is known also that the inequality $\mathrm{compl}(X) \le 66$ holds for any isolated log-canonical three-dimensional singularity [8].

. Now we consider cases when $S \neq \mathbb{P}^2$. In many cases, S is a cone over a rational normal curve.

Cases when $S \simeq \mathbb{P}(1, 1, 2)$ (quadratic cone). In cases: $[2, 3, 7, 2r]$, $r \in \{4, 5, 8, 10, 11, 13, 16, 17, 19, 20\}$, $[2, 3, 8, r]$, $r \in \{11, 13, 17, 19, 23\}$, $[2, 4, 5, r]$, $r \in \{7, 9, 11, 13, 17, 19\}$, $[2, 4, 7, 9]$, $[2, 3, 10, r]$, $r \in \{11, 13\}$, $[2, 5, 6, 7]$ and $[2, 4, 7, 8]$, $\varDelta$ has a component $\varDelta_1 \equiv 2H$ with the coefficient $\ge 6/7$. By Lemma 5.14, for any regular complement $\varDelta^+$ we have $\varDelta^+ \ge \varDelta_1$. This yields a contradiction with $\varDelta^+ \equiv -K_S \equiv 2H$.

Similarly, in cases $[2, 4, 5, 4r]$, $r = 2, 3, 4$ and $[2, 3, 8, 20]$, $\varDelta$ has a component $\varDelta_1$ with the coefficient $4/5$. Again $\varDelta^+ \ge \frac{5}{6}\varDelta_1$. In case $[2, 4, 5, 16]$ this gives us a contradiction. In cases $[2, 4, 5, 12]$ and $[2, 3, 8, 20]$, we obtain $\mathrm{Supp}(\varDelta^+) = \mathrm{Supp}(\varDelta)$. By Corollary 5.6.1, $K_S + \varDelta^+$ is klt. Finally, in case $[2, 4, 5, 8]$ we have only one possibility with $\mathrm{Supp}(\varDelta^+) \neq \mathrm{Supp}(\varDelta)$: $\varDelta^+ = \frac{5}{6}C_4 + \frac{1}{2}L_3 + \frac{1}{6}M$, where $M \equiv H$, $M \neq L_3$. Since $C_4 \cap L_3 \cap M = \varnothing$, we may apply Lemma 5.15. Here $K_S + M$ is plt by Lemma 5.16 below.

Lemma 5.16. *Let (S, o) be a germ of a surface singularity of type $\frac{1}{n}(1, 1)$ and let M be a germ of a smooth curve passing through o. Then $K_S + M$ is plt at o.*

Proof. Let $\sigma\colon \tilde{S} \to S$ be the minimal resolution, let E be the (irreducible) exceptional divisor and let $\tilde{M}$ be the proper transform of M. Write $\sigma^*(K_S + M) = K_{\tilde{S}} + \tilde{M} + \alpha E$. Then σ is a log-resolution and it is sufficient to show that $\alpha < 1$. By Adjunction, $0 = (K_{\tilde{S}} + E)\cdot E + \tilde{M}\cdot E + (\alpha-1)E^2 = -2+1-(\alpha-1)n$. Thus $\alpha = 1 - 1/n$. $\square$

Cases when $S \simeq \mathbb{P}(1,1,3)$ *(rational cubic cone).* Almost all cases can be treated as above because Δ has a component with the coefficient $4/5$ or $\geq 6/7$. The only non-trivial case is $[2,3,9,9]$. Then $\Delta' \equiv \frac{1}{2}H$, where H is as in 5.12. Therefore C_9 is not a component of Δ'. If $\mathrm{Supp}(\Delta') \not\subset \sum L_i$, then we have only one possibility $\Delta' = \frac{1}{6}M$, where $M \equiv 3H$, or $\Delta' = \sum \alpha_i M_i$, where M_i's are generators of the cone and $\sum \alpha_i = 1/2$. In both cases $K_S + \Delta^+$ is klt by Lemma 5.15.

Case $[2,3,8,8r]$, $r = 1,2$. From $\Delta_{r=1} \leq \Delta_{r=2}$ we may assume that $r = 1$, i.e., $\Delta = \frac{2}{3}C_8$. Then $\Delta' \equiv \frac{2}{3}H$. If the coefficient of C_8 in Δ^+ is bigger than that in Δ, then there is only one possibility $\Delta^+ = \frac{3}{4}C_8$. Clearly, this complement is klt. Thus we may assume that $\Delta^+ = \frac{2}{3}C_8 + \Delta'$, where C_8 is not a component of Δ'. Note that S is isomorphic to a projective cone in $\mathbb{P}^5$ over a rational normal curve of degree 4. If all components of Δ' are generators, then we can write $\Delta' = \sum \alpha_i M_i$, where $\sum \alpha_i = 2/3$. By Lemma 5.15 and Lemma 5.16, $K_S + \frac{2}{3}C_8 + \Delta'$ is klt at the vertex in this case. Assume that $K_S + \frac{2}{3}C_8 + \Delta'$ is not klt at some point P (outside of the vertex). Then $P \in C_8$ and there is exactly one component, say M_1 of Δ' passing through P. By assumption, C_8 and M_1 at P cannot intersect each other transversally. Since $C_8 \cdot M_1 = 8H^2 = 2$, C_8 and M_1 have simple tangency at P. Using $\alpha_1 \leq 2/3$, we can easily check that $K_S + \frac{2}{3}C_8 + \alpha_1 M_1$ is klt at P. Finally, assume that $\Delta' = \sum \alpha_i M_i$ and M_1 is not a generator of the cone. Then $\alpha_1 \geq 1/6$ and $M_1 \sim kH$, where $k \geq 4$. This yields $k = 4$ and $\Delta' = \frac{1}{6}M_1$. Since M_1 is irreducible, it does not contain the vertex of the cone. By Lemma 5.15, $K_S + \frac{2}{3}C_8 + \frac{1}{6}M_1$ is klt.

Cases $[2,3,7,35]$ *and* $[3,4,4,5]$. The only regular complements are $\Delta^+ = \frac{1}{2}C_7 + \frac{2}{3}L_1 + \frac{5}{6}L_3$ and $\Delta^+ = \frac{2}{3}L_1 + \frac{5}{6}C_4$, respectively. They are klt (see Corollary 5.6.1).

Case $[2,3,10,10]$. Then $\Delta' \equiv \frac{1}{3}H$. If the coefficient α of C_{10} in Δ^+ is bigger than that in Δ, then $7/10 \geq \alpha > 2/3$ and $12\alpha \in \mathbb{Z}$. This is impossible. As in the case $[2,3,8,8]$ there is only one possibility: $\Delta^+ = \frac{2}{3}C_{10} + \sum \alpha_i M_i$, where $\sum \alpha_i = 1/3$. By Lemma 5.15 this complement is klt.

Case $[2,3,11,11]$. Then $\Delta' \equiv \frac{1}{6}H$. The only possibility for regular complement with $\mathrm{Supp}(\Delta^+) \neq \mathrm{Supp}(\Delta)$ is $\Delta^+ = \frac{1}{2}C_{11} + \frac{2}{3}L_1 + \frac{1}{6}M$, where $M := \{z = cx\}$.

By Lemma 5.16, $K_S + M$ is plt. We have to check only that $K_S + \varDelta^+$ is klt near M. This follows by Lemma 5.15 because $M \cap C_{11} \cap L_1 = \varnothing$.

Cases when $S \simeq \mathbb{P}(3, 2, 1)$ *(Gorenstein del Pezzo surface of degree* 6*).* All cases are exceptional because we have a component C_6 of $\varDelta$ with the coefficient $\geq 6/7$.

Case $[2, 3, 7, 21]$. Then $\varDelta' \equiv \frac{1}{2}H$. Since $K_S + \frac{2}{3}C_{21}$ is ample, the coefficient of C_{21} in $\varDelta^+$ is $1/2$. If $\operatorname{Supp}(\varDelta^+) \not\subset \varGamma$, then $\varDelta^+ = \frac{1}{2}C_{21} + \frac{1}{6}M$, where $M := \{x_2 = cx_3^3\}$, $c \neq 0$. The curve M contains the point $(1, 0, 0)$ of type $\frac{1}{7}(1, 2)$. Taking into account that $(K_S + M) \cdot M = -8/7$ and by Lemma 5.17 below, $K_S + M$ is plt. Finally, by Lemma 5.15, $K_S + \varDelta^+$ is klt.

Lemma 5.17. *Let* S *be a projective surface with only log terminal singularities and let* M *be an irreducible curve on* S*. Assume that* M *contains singular points* $P_1, \dots, P_r$ *of* S *of types* $\frac{1}{m_1}(1, q_1), \dots, \frac{1}{m_r}(1, q_r)$*, where* $\gcd(m_i, q_i) = 1$*,* $\forall i$*. If*

$$(K_S + M) \cdot M \leq -2 + \sum (1 - 1/m_i), \tag{5.IV}$$

then $K_S + M$ *is plt near* M*.*

Proof. Let $\Phi \subset M$ be the set of points where $K_S + M$ is not plt. Let $\nu\colon \widehat{S} \to S$ be a birational morphism which is a log resolution over Φ and an isomorphism outside of Φ. Write

$$\nu^*(K_S + M) = K_{\widehat{S}} + \widehat{M} + \sum a_i E_i,$$

where $\widehat{M}$ is the proper transform of M and $\sum a_i E_i$ is the exceptional divisor. Then $\widehat{M}$ is smooth and by Adjunction

$$\left(K_{\widehat{S}} + \widehat{M} + \sum a_i E_i\right)\Big|_{\widehat{M}} = K_{\widehat{M}} + \operatorname{Diff}_{\widehat{M}}\left(\sum a_i E_i\right).$$

We can write

$$\operatorname{Diff}_{\widehat{M}}\left(\sum a_i E_i\right) = \sum_{P' \in \nu^{-1}(\Phi) \cap \widehat{M}} a' P' + \sum_{P' \notin \nu^{-1}(\Phi) \cap \widehat{M}} b' P',$$

If $P \notin \Phi$, then by construction, ν is an isomorphism over P and by [12, 16.6], $(S, M) \simeq_{an} (\mathbb{C}^2, \{x = 0\})/\mathbb{Z}_m(1, q)$, $\gcd(m, q) = 1$ and $b' = 1 - 1/m$.

If $P \in \Phi$ then by Connectedness Lemma [20, 5.7], $\widehat{M} + \sum_{a_i \geq 1} E_i$ is connected near $\nu^{-1}(P)$. Thus the coefficient a' of $\operatorname{Diff}_{\widehat{M}}(\sum a_i E_i)$ at $P' \in \nu^{-1}(P)$ is ≥ 1. Now we have $\operatorname{Diff}_{\widehat{M}}(\sum a_i E_i) \geq \sum_{i=1}^r (1 - 1/m_i) P_i'$, where $\nu(P_i') = P_i$. Combining this with (5.IV) and $\deg K_{\widehat{M}} \geq -2$, we obtain $\nu^{-1}(P_i) = \{P_i'\}$ and $\operatorname{Diff}_{\widehat{M}}(\sum a_i E_i) = \sum_{i=1}^r (1 - 1/m_i) P_i'$. In particular,

$$\left\lfloor \operatorname{Diff}_{\widehat{M}}\left(\sum a_i E_i\right)\right\rfloor = 0$$

and $\Phi = \varnothing$. $\square$

Cases $[2,3,7,14r]$, $r=1,2$. Since $\Delta_{r=1} \le \Delta_{r=2}$, it is sufficient to consider only case $r=1$. We have $\Delta' \equiv \frac{2}{3}H$. If C_{14} and L_i's are the only components of Δ^+, then $K_S+\Delta^+$ is klt by 5.6.1. Assume that there is a component $M \ne L_i, C_{14}$. Then $M \in |rH|$, for some r. It is clear that $-(K_S+\frac{1}{2}L_3+\frac{2}{3}C_{14}+\frac{1}{6}M)$ is nef. This gives us $10-\frac{2}{3}\cdot 14-\frac{1}{6}r \ge 0$ and $r \le 4$. Since $M \ne L_3$, $r>1$. Further, M contains the point $\{y=z=0\}$ of type $\frac{1}{7}(2,1)$. By Lemma 5.17, K_S+M is plt. Then $K_S+\Delta^+$ is klt by Lemma 5.15.

5.18. **Singularities which do not satisfy** $(\star)$**.** Now we consider singularities which do not satisfy the condition of $(\star)$. We will see that all of them are not analytically $\mathbb{Q}$-factorial (see 5.18.3). Recall that we assumed that $S \subset \mathbb{P}(\overline{\mathbf{p}})$ with normalized $\overline{\mathbf{p}}$. By H denote the class of Weil divisors such that $\mathcal{O}_S(H) = \mathcal{O}_S(1)$. By Lemma 5.9,

$$H^2 = \overline{w}/\overline{p}_1\cdots\overline{p}_4, \qquad K_S \sim \left(\overline{w}-\sum \overline{p}_i\right) H,$$

where $\overline{w} = \overline{a}_1\overline{p}_1 = \cdots = \overline{a}_4\overline{p}_4$. Taking into account 5.10 and 5.5.1 we obtain Table 2 and additionally cases $[3,3,4,4]$ and $[2,4,5,5]$ below.

5.18.1. *Case* $[3,3,4,4]$. Then $S=\{y_1^3+y_2^3+y_3^4+y_4^4=0\} \subset \mathbb{P}(4,4,3,3)$ and $\Delta=0$. We claim that the singularity is not exceptional. Indeed, consider the curve $M := S \cap \{y_3=\omega y_4\}$, where $\omega^4=-1$. We have $K_S+\frac{2}{3}M \sim_{\mathbb{Q}} 0$. It is sufficient to show that $K_S+\frac{2}{3}M$ is not klt (see Proposition 2.3). Indeed, $M=\{y_3-\omega y_4 = y_1^3+y_2^3=0\}$ has three components passing through one point $(0,0,\omega,1)$. This point is singular of type $\frac{1}{3}(1,1)$. By blowing-up it we obtain an exceptional divisor E with discrepancy

$$a\left(E,\frac{2}{3}M\right) = a(E,0)-\frac{2}{3}\Big(m_1+m_2+m_3\Big) = -\frac{1}{3}-\frac{2}{3}\Big(m_1+m_2+m_3\Big),$$

where m_1, m_2, m_3 are multiplicities of components of M. Clearly, $m_1,m_2,m_3 \ge 1/3$. Therefore $a(E,M) \le -1$ and $K_S+\mathrm{Diff}_S(\frac{2}{3}M)$ is not klt.

5.18.2. *Case* $[2,4,5,5]$. Then $S=\{y_1^2+y_2^2+y_3^5+y_4^5=0\} \subset \mathbb{P}(5,5,2,2)$ and $\Delta=\frac{1}{2}\Gamma_2$. As in case $[3,3,4,4]$ we can take $\Delta^+=\frac{1}{2}\Gamma_2+\frac{3}{4}M$, where $M:=\{y_4+y_3=0\}\cap S$. Then $K_S+\Delta^+$ is not klt at $(0,0,1,-1)$. This implies that the singularity is not exceptional.

. Now we prove case by case that all singularities in Table 2 are exceptional. Assuming that there exists a regular non-klt n-complement $K_S + \varDelta^+$, we derive a contradiction or prove that $K_S + \varDelta^+$ is klt (see Proposition 2.3). Set $\varDelta' := \varDelta^+ - \varDelta$. By Corollary 5.6.1 we may assume either $\operatorname{Supp}(\varDelta^+) \not\subset \varGamma$ or $\operatorname{Supp}(\varDelta^+) \subset \varGamma$ and $\lfloor \varDelta^+ \rfloor \neq 0$. By [16, Proposition 2] (or [17, 4.9]) and (i) of 5.6, for any n-complement $K_S + \varDelta^+$ there exists an n-complement $K_{\mathbb{P}} + D$ such that $D|_S = \varDelta^+$. So we can write

$$\varDelta^+ = \sum \frac{k_i}{n} \varGamma_i + \sum \frac{s_j}{n} M_j, \tag{5.V}$$

where $k_i, s_j \in \mathbb{N} \cup \{0\}$, M_j's are effective (not necessarily irreducible) curves and $M_j \sim m_j H$. We may assume that $M_j \neq \varGamma_i$, $\forall i, j$. By 5.6.1 one of the following holds $\sum \frac{s_j}{n} M_j \neq 0$ or $\lfloor \sum \frac{k_i}{n} \varGamma_i \rfloor \neq 0$. By the construction, $\sum \frac{s_j}{n} M_j \leq \varDelta'$.

Cases $[2, 3, 8, 18]$, $[2, 4, 6, 9]$ *and* $[2, 3, 9, 12]$. Then $\varDelta' \equiv \frac{1}{12} H$, $\frac{1}{6} H$ and $\frac{1}{6} H$, respectively. Here $\operatorname{Supp}(\varDelta^+) \subset \varGamma$. It is easy to check that $\lfloor \varDelta^+ \rfloor = 0$. This implies that $K_S + \varDelta^+$ is klt by Corollary 5.6.1.

Case $[2, 4, 6, 10]$. By 5.14, $\varDelta^+ \geq \frac{1}{2} \varGamma_2 + \frac{2}{3} \varGamma_3 + \frac{5}{6} \varGamma_4$. Since $\varDelta^+ \equiv 2H$, $2 \geq \frac{1}{2} + \frac{2}{3} + \frac{5}{6} = 2$. This yields the equality $\varDelta^+ = \frac{1}{2} \varGamma_2 + \frac{2}{3} \varGamma_3 + \frac{5}{6} \varGamma_4$. This complement is klt by 5.6.1. Similarly, we can argue in cases $[3, 3, 4, 10]$, $[2, 3, 9, 10]$ and $[2, 3, 10, 12]$.

Cases $[2, 3, 8, 3r]$, $r \in \{3, 5, 7\}$, $[2, 3, 9, 2r]$, $r \in \{7, 8\}$. By $\varDelta_{r=3} \leq \varDelta_{r=5} \leq \varDelta_{r=7}$ it is sufficient to consider only case $[2, 3, 8, 9]$. Then $\varDelta' \equiv \frac{5}{12} H$. Let M_j be as in (5.V). Then $\frac{1}{n} M_j \leq \varDelta'$. Hence, $m_j / n \leq 5/12$. This yields $n = 6$ and $m_j = 2$. But then $\varDelta^+ \geq \frac{5}{6} \varGamma_3 + \frac{2}{3} \varGamma_4 + \frac{1}{6} M$, so $4 \geq \frac{5}{6} \cdot 3 + \frac{2}{3} \cdot 2 + \frac{1}{6} \cdot 2 = 25/6$, a contradiction.

Case $[3, 3, 5, 5]$. Then $\varDelta^+ \equiv H$. Thus $n\varDelta^+$ is given by some polynomial $p \in H^0(S, \mathcal{O}_S(n))$, where $n \in \{1, 2, 3, 4, 6\}$. On the other hand, $H^0(S, \mathcal{O}_S(n)) \neq 0$ for $n = 3$ or 6. We obtain two cases: $\varDelta^+ = \frac{1}{3} M$, where M is given by $c_3 y_3 + c_4 y_4 = 0$, or $\varDelta^+ = \frac{1}{6} M + \frac{1}{6} M'$, where M and M' are given by $c_3 y_3 + c_4 y_4 = 0$ and $c'_3 y_3 + c'_4 y_4 = 0$, respectively. Consider for example, the first case. If $c_3^5 + c_4^5 \neq 0$, then M is irreducible. If $c_3^5 + c_4^5 = 0$, then M has exactly three irreducible components. By Lemma 5.15 it is sufficient to show that $K_S + M^{(i)}$ is plt for any irreducible component $M^{(i)}$ of M. Note that M contains three singular points $\varGamma_3 \cap \varGamma_4$ of type $\frac{1}{5}(1, 1)$. If M is irreducible, then $(K_S + M) \cdot M = 6H^2 = 2/5$ and Lemma 5.17 give us that $K_S + M$ is plt. If M is not irreducible, then all the irreducible components $M^{(i)} \subset M$ are smooth. By Lemma 5.16, $K_S + M^{(i)}$ is plt.

Case $[3,3,4,8]$. Then $\Delta' \equiv \frac{1}{2}H$. Obviously, $\lfloor \Delta^+ \rfloor = 0$. Hence, $\mathrm{Supp}(\Delta^+) \not\subset \Gamma$. We have only one possibility: $\Delta^+ = \frac{1}{2}\Gamma_4 + \frac{1}{6}M$, where $M := S \cap \{y_3 = cy_4\}$. This curve is irreducible if $1 + c^4 \neq 0$ and has exactly three irreducible components if $1+c^4 = 0$. By Lemma 5.15 it is sufficient to show that $K_S+\Gamma$ is plt for any irreducible component of M. Note that M contains three singular points $\Gamma_3 \cap \Gamma_4$ of type $\frac{1}{4}(1,1)$ and $(K_S + M) \cdot M = 3H^2 = 1/4$. If M is irreducible, then by Lemma 5.17, $K_S + M$ is plt. If M is not irreducible, then all the irreducible components $M^{(i)} \subset M$ are smooth by Lemma 5.16. Thus $K_S + M^{(i)}$ is plt.

Case $[2,4,7,7]$. Then $\Delta' \equiv \frac{1}{2}H$. It is easy to see that the coefficient of Γ_2 in Δ^+ cannot be greater than $1/2$. We have $\Delta^+ = \frac{1}{2}\Gamma_2 + \frac{1}{4}M$, where M is cut out on S by $y_3 = cy_4 = 0$. If $1 + c^7 \neq 0$, then M is irreducible. If $1 + c^7 = 0$, then M has exactly two irreducible components. By Lemma 5.15 it is sufficient to show that $K_S + M^{(i)}$ is plt for any irreducible component of M. Note that M contains two singular points $\Gamma_3 \cap \Gamma_4$ of type $\frac{1}{7}(1,1)$. It is easy to compute also $(K_S + M) \cdot M = -4H^2 = -2/7$. If M is irreducible, then by Lemma 5.17, $K_S + M$ is plt. If M is not irreducible, then all the irreducible components $M^{(i)} \subset M$ are smooth by Lemma 5.16. Hence, $K_S + M^{(i)}$ is plt.

Case $[3,3,4,6]$. Then $\Delta' \equiv \frac{1}{2}H$. First we claim that $K_S + \Delta^+$ is klt outside of Γ_3. Assume the opposite. Then $K_S + \Delta'$ is not klt and $K_S + \Delta' + \Gamma_4$ is not plt. Write $\Delta' = \alpha\Gamma_4 + \Delta''$, where $\alpha \geq 0$ and Γ_4 is not a component of $\mathrm{Supp}(\Delta'')$. By Connectedness Lemma [20, 5.7], $K_S + \Delta'' + \Gamma_4$ is not plt near Γ_4. Further, $\mathrm{Diff}_{\Gamma_4}(0) = \frac{1}{2}P_1 + \frac{1}{2}P_2 + \frac{1}{2}P_3$, where $\{P_1, P_2, P_3\} = \Gamma_3 \cap \Gamma_4 = \mathrm{Sing}(S)$. On the other hand, $\Delta'' \cdot \Gamma_4 \leq \Delta' \cdot \Gamma_4 = 1/4$. This yields $\lfloor \mathrm{Diff}_{\Gamma_4}(\Delta'') \rfloor = 0$. By Inversion of Adjunction, $K_S + \Gamma_4 + \Delta''$ is plt near Γ_4, a contradiction.

Now we claim that $K_S + \Delta^+$ is klt outside of $\mathrm{Sing}(S)$. As above, $\mathrm{Diff}_{\Gamma_3}(0) = \frac{1}{2}P_1 + \frac{1}{2}P_2 + \frac{1}{2}P_3$ and $\Delta' \cdot \Gamma_3 = 3/4$. If $K_S + \Delta^+$ is not klt at some point $P \notin \mathrm{Sing}(S)$, then $K_S + \Gamma_3 + \Delta'$ is not plt at P. Thus $\mathrm{Diff}_{\Gamma_3}(\Delta') \geq \frac{1}{2}P_1 + \frac{1}{2}P_2 + \frac{1}{2}P_3 + P$, a contradiction.

Further, fix a point $P_1 \in \mathrm{Sing}(S)$ (of type A_1), say $P_1 = (1,-1,0,0)$ and let M be as in (5.V). We may assume that $P_1 \in M$ and $K_S + \Delta^+$ is not klt at P_1. Since $\Delta' \equiv \frac{1}{2}H$, there are three cases:

(i) $\Delta^+ = \frac{1}{2}\Gamma_3 + \frac{1}{4}M$, where $M := \{y_1 + y_2 = cy_4^2\} \cap S$;
(ii) $\Delta^+ = \frac{1}{2}\Gamma_3 + \frac{1}{6}\Gamma_4 + \frac{1}{6}M$, where $M := \{y_1 + y_2 = cy_4^2\} \cap S$;
(iii) $\Delta^+ = \frac{1}{2}\Gamma_3 + \frac{1}{6}M$, where $M := \{y_3 = y_4(c_1y_1 + c_2y_2 + c_4y_4^2)\} \cap S$.

By Lemma 5.15 and Lemma 5.16 it is sufficient to show that M is either smooth at P_1 or M has two smooth analytic components at P_1. Indeed, in cases (i) and (ii), M is given by $y_3^2+3cy_4^2y_1^2-3c^2y_4^4y_1+(c^3+1)y_4^6 = 0$ in $\mathbb{P}_{y_1,y_3,y_4}(2,3,1)$. The local equation of M near $(1,0,0)$ is $x^2 + 3cy^2 - 3c^2y^4 + (1 + c^3)y^6 = 0$ in

$\mathbb{C}^2/\mathbb{Z}_2(1,1)$. Thus, in cases (i) and (ii), M has exactly two components which are smooth curves. Similarly, in case (iii), M is smooth at P_1.

Case $[2,4,6,8]$. Then $\Delta' \equiv \frac{1}{6}H$. It is easy to compute that $\lfloor \Delta^+ \rfloor = 0$. Let M_1 be as in (5.V). We have $\frac{1}{6}m_1 \le 1/6$. Hence, $\Delta' = \frac{1}{6}M_1$, where $M_1 := \{y_4 = cy_2\} \cap S$, $c \ne 0$. If $1 + c^4 \ne 0$, then $M_1 \simeq \mathbb{P}^1$. If $1 + c^4 = 0$, then M_1 has exactly two components $\simeq \mathbb{P}^1$. Further, $\mathrm{Diff}_{\varGamma_4}(0) = \frac{1}{2}P_1 + \frac{1}{2}P_2$, where $\{P_1, P_2\} = \varGamma_2 \cap \varGamma_4$. On the other hand, $\Delta' \cdot \varGamma_4 = 1/6$ yields $\lfloor \mathrm{Diff}_{\varGamma_4}(\Delta') \rfloor = 0$. By Inversion of Adjunction, $K_S + \varGamma_4 + \Delta'$ is plt near $\varGamma_4$. Thus $K_S + \Delta^+$ is klt near $\varGamma_4$. Outside of $\varGamma_4$ the surface S is smooth and M_1 has at most two irreducible components (and they both are smooth). By Lemma 5.15, $K_S + \Delta^+$ is klt.

Case $[2,4,6,6]$. Then $\Delta' \equiv \frac{1}{2}H$. Clearly, the coefficient of $\varGamma_2$ in Δ^+ is $1/2$. Thus $\varGamma_2$ is not a component of $\mathrm{Supp}(\Delta')$. First we claim that $K_S + \Delta^+$ is klt near $\varGamma_2$. Since $\Delta' \cdot \varGamma_2 = 1$ and because $\varGamma_2$ is contained in the smooth locus of S, we have that $\lfloor \mathrm{Diff}_{\varGamma_2}(\Delta') \rfloor$ is reduced. By Inversion of Adjunction, $K_S + \varGamma_2 + \Delta'$ is lc near $\varGamma_2$. Then $K_S + \frac{1}{2}\varGamma_2 + \Delta'$ is klt near $\varGamma_2$. This proves our claim.

Since $K_S + \Delta^+$ is klt near $\varGamma_2$, $\lfloor \Delta^+ \rfloor = 0$. Now we have to show only that $K_S + \Delta^+$ is klt outside of $\varGamma_2$. Assume the opposite. Then $K_S + \Delta'$ is not klt. Write $\Delta' = \alpha\varGamma_3 + \Delta''$, where $\alpha \ge 0$ and $\varGamma_3$ is not a component of $\mathrm{Supp}(\Delta')$. Then $-(K_S + \varGamma_3 + \Delta'')$ is ample. By Connectedness Lemma [20, 5.7], $K_S + \varGamma_3 + \Delta''$ is not plt near $\varGamma_3$ and klt outside of $\varGamma_3$. Note that $\varGamma_3$ contains exactly two singular points $\{P_1, P_2\} = \varGamma_3 \cap \varGamma_4$ and they are of type $\frac{1}{3}(1,1)$. Thus $\mathrm{Diff}_{\varGamma_3}(0) = \frac{2}{3}P_1 + \frac{2}{3}P_2$. On the other hand, $\Delta'' \cdot \varGamma_3 \le \Delta' \cdot \varGamma_3 = 1/3$. From this we have that $\lceil \mathrm{Diff}_{\varGamma_3}(\Delta'') \rceil$ is reduced, i.e., $K_{\varGamma_3} + \mathrm{Diff}_{\varGamma_3}(\Delta'')$ is lc. Again by Inversion of Adjunction (see [12, 17.7]), $K_S + \varGamma_3 + \Delta''$ is lc near $\varGamma_3$. Therefore, $K_S + \alpha\varGamma_3 + \Delta''$ is klt near $\varGamma_3$, a contradiction.

Case $[2,4,5,10]$. Then $\Delta' \equiv \frac{1}{2}H$. There are only two cases: $\Delta^+ = \frac{1}{2}\varGamma_2 + \frac{1}{4}M$ or $\Delta^+ = \frac{1}{2}\varGamma_2 + \frac{1}{6}\varGamma_4 + \frac{1}{6}M$, where M is cut out on S by $y_3 = \alpha y_4^2$. It is easy to see that M is isomorphic to a (reduced) conic. So M has at most two irreducible components and they are smooth. By Lemma 5.15, $K_X + \Delta^+$ is klt outside of $\mathrm{Sing}(S) = \varGamma_3 \cap \varGamma_4$. Near $\mathrm{Sing}(S)$ we have $\Delta^+ = \Delta'$. It is sufficient to show that $K_S + \varGamma_4 + \Delta'$ is plt near $\varGamma_4$. Indeed, $\mathrm{Diff}_{\varGamma_4}(0) = \frac{4}{5}P_1 + \frac{4}{5}P_2$ and $\Delta' \cdot \varGamma_4 = 1/10 < 1/5$. This yields $\lfloor \mathrm{Diff}_{\varGamma_4}(\Delta') \rfloor = 0$. By Inversion of Adjunction, $K_S + \varGamma_4 + \Delta'$ is plt near $\varGamma_4$.

Cases $[2,5,5,6]$, $[2,4,5,15]$, $[2,5,5,8]$. It is sufficient to consider only case $[2,5,5,6]$ (when Δ is smaller). Then $\Delta' \equiv \frac{2}{3}H$. Let γ be the coefficient of $\varGamma_4$ in Δ^+. Assume that $\gamma > 2/3$. Since $\varGamma_4 \sim 5H$, we have $\gamma - 2/3 \le 5/6$. Taking into account that $\gamma = k/n$, $k \in \mathbb{N}$, we obtain $\gamma = 3/4$ and $n = 4$. This means that $4(\Delta^+ - \frac{3}{4}\varGamma_4)$, is an integral effective divisor on S. On the

other hand, $4\Delta' \equiv H$, a contradiction. Therefore we may assume that $\gamma = 2/3$ and $n = 3$ or 6. By our assumption Γ_4 is not a component of Δ'. First, we claim that $K_S + \Gamma_4 + \Delta'$ is plt near Γ_4. By Lemma 5.11, S has on Γ_4 five points $P_1, \dots, P_5$ of type A_1. Thus, $\mathrm{Diff}_{\Gamma_4}(0) = \frac{1}{2}\sum P_i$. On the other hand, $\Delta' \cdot \Gamma_4 = \frac{10}{3}H^2 = 1/3$. Therefore, $\lfloor \mathrm{Diff}_{\Gamma_4}(\Delta') \rfloor = 0$. By Inversion of Adjunction, $K_S + \Gamma_4 + \Delta'$ is plt near Γ_4. Assume that $K_S + \Delta^+$ is not klt outside of Δ_4. Then $K_S + \Delta'$ is not klt. Since $-(K_S + \Gamma_2 + \Delta')$ is ample, by Connectedness Lemma [20, 5.7], $K_S + \Gamma_2 + \Delta'$ is not plt near Γ_2. Hence, $K_{\Gamma_2} + \mathrm{Diff}_{\Gamma_2}(\Delta')$ is not klt (i.e. $\lfloor \mathrm{Diff}_{\Gamma_2}(\Delta') \rfloor \neq 0$). On the other hand, as above, $\mathrm{Diff}_{\Gamma_2}(0) = \frac{4}{5}Q_1 + \frac{4}{5}Q_2$ and $\Delta' \cdot \Gamma_2 = 2/15 < 1/5$. So $\lfloor \mathrm{Diff}_{\Gamma_2}(\Delta') \rfloor \neq 0$, a contradiction.

Case $[2, 3, 8, 12]$. Then $\Delta' \equiv \frac{1}{2}H$. If $\mathrm{Supp}(\Delta^+) \subset \Gamma$, then $\lfloor \Delta^+ \rfloor = 0$ and $K_S + \Delta^+$ is klt by 5.6.1. If $\mathrm{Supp}(\Delta^+) \not\subset \Gamma$, then the only possibility is $\Delta^+ = \frac{1}{2}\Gamma_3 + \frac{1}{6}M$, where $M = \{y_3 = cy_4^3\} \cap S$, $c \neq 0$. It is easy to see that $M \simeq \{x^2 + y^3 + (1+c^4)z^6 = 0\} \subset \mathbb{P}(3, 2, 1)$. This curve is irreducible and $p_a(M) = 1$. If $1 + c^4 \neq 0$, then M is smooth. If $1 + c^4 = 0$, then M has a simple cusp at $P = \{(0, 0, c, 1)\}$. By Lemma 5.11, S is smooth at P. Further, S has only singularities of type A_1 and $\frac{1}{3}(1, 1)$ (see 5.11). Lemma 5.16 and Lemma 5.15 give us that $K_S + \Delta^+$ is klt outside of P. But at P we have $\Delta^+ = \frac{1}{6}M$, where M has a simple cusp at P. At this point $K_S + \frac{1}{6}M$ is also klt (see, e.g., [11, 8.14]).

Table 1. Case: $\mathrm{Pic}(S) = \mathbb{Z}$

No.	a_1	a_2	a_3	a_4	S	$\mathrm{Diff}_S(0)$
1	2	3	7	r	$\mathbb{P}^2$	$\frac{1}{2}L_1 + \frac{2}{3}L_2 + \frac{6}{7}L_3 + \frac{r-1}{r}C_1$
	$r \in \{11, 13, 17, 19, 23, 25, 29, 31, 37, 41\}$					
2	2	3	11	13	$\mathbb{P}^2$	$\frac{1}{2}L_1 + \frac{2}{3}L_2 + \frac{10}{11}L_3 + \frac{12}{13}C_1$
3	2	4	5	$2r$	$\mathbb{P}^2$	$\frac{1}{2}L_2 + \frac{4}{5}C_2 + \frac{r-1}{r}L_3$
	$r \in \{3, 7, 9\}$					
4	2	4	6	r	$\mathbb{P}^2$	$\frac{1}{2}L_2 + \frac{2}{3}L_3 + \frac{r-1}{r}C_2$
	$r \in \{7, 11\}$					
5	2	3	8	$2r$	$\mathbb{P}^2$	$\frac{2}{3}C_2 + \frac{3}{4}L_2 + \frac{r-1}{r}L_3$
	$r \in \{5, 7, 11\}$					
6	2	3	10	14	$\mathbb{P}^2$	$\frac{2}{3}C_2 + \frac{4}{5}L_2 + \frac{6}{7}L_3$
7	2	3	9	15	$\mathbb{P}^2$	$\frac{1}{2}C_3 + \frac{2}{3}L_2 + \frac{4}{5}L_3$
8	3	3	4	9	$\mathbb{P}^2$	$\frac{3}{4}C_3 + \frac{2}{3}L_3$
9	3	3	5	6	$\mathbb{P}^2$	$\frac{4}{5}C_3 + \frac{1}{2}L_3$
10	3	4	4	4	$\mathbb{P}^2$	$\frac{2}{3}C_4$
11	2	5	5	5	$\mathbb{P}^2$	$\frac{1}{2}C_5$
12	2	3	7	$2r$	$\mathbb{P}(1,2,1)$	$\frac{2}{3}C_2 + \frac{6}{7}L_2 + \frac{r-1}{r}L_3$
	$r \in \{4, 5, 8, 10, 11, 13, 16, 17, 19, 20\}$					
13	2	3	8	r	$\mathbb{P}(1,1,2)$	$\frac{2}{3}C_2 + \frac{3}{4}L_2 + \frac{r-1}{r}L_3$
	$r \in \{11, 13, 17, 19, 23\}$					
14	2	3	8	20	$\mathbb{P}(2,1,1)$	$\frac{2}{3}C_4 + \frac{1}{2}L_2 + \frac{4}{5}L_3$
15	2	4	5	r	$\mathbb{P}(1,1,2)$	$\frac{1}{2}L_2 + \frac{4}{5}L_3 + \frac{r-1}{r}C_2$
	$r \in \{7, 9, 11, 13, 17, 19\}$					
16	2	4	5	$4r$	$\mathbb{P}(2,1,1)$	$\frac{4}{5}C_4 + \frac{r-1}{r}L_3$
	$r \in \{2, 3, 4\}$					
17	2	4	7	8	$\mathbb{P}(2,1,1)$	$\frac{6}{7}C_4 + \frac{1}{2}L_3$
18	2	4	7	9	$\mathbb{P}(1,1,2)$	$\frac{1}{2}L_2 + \frac{6}{7}L_3 + \frac{8}{9}C_2$
19	2	3	10	r	$\mathbb{P}(1,1,2)$	$\frac{2}{3}C_2 + \frac{4}{5}L_2 + \frac{r-1}{r}L_3$
	$r \in \{11, 13\}$					
20	2	5	6	7	$\mathbb{P}(1,1,2)$	$\frac{4}{5}C_2 + \frac{2}{3}L_2 + \frac{6}{7}L_3$
21	2	3	7	$3r$	$\mathbb{P}(3,1,1)$	$\frac{1}{2}L_1 + \frac{6}{7}C_3 + \frac{r-1}{r}L_3$
	$r \in \{3, 5, 9, 11, 13\}$					
22	2	3	9	9	$\mathbb{P}(3,1,1)$	$\frac{1}{2}C_9$
23	2	3	9	r	$\mathbb{P}(3,1,1)$	$\frac{1}{2}L_1 + \frac{2}{3}L_3 + \frac{r-1}{r}C_3$
	$r \in \{11, 13, 17\}$					
24	3	3	4	r	$\mathbb{P}(1,1,3)$	$\frac{3}{4}L_3 + \frac{r-1}{r}C_3$
	$r \in \{5, 7, 11\}$					
25	3	3	5	7	$\mathbb{P}(1,1,3)$	$\frac{4}{5}L_3 + \frac{6}{7}C_3$
26	2	5	6	6	$\mathbb{P}(3,1,1)$	$\frac{4}{5}C_6$

No.	a_1	a_2	a_3	a_4	S	$\mathrm{Diff}_S(0)$
27	2	3	8	$8r$	$\mathbb{P}(4,1,1)$	$\frac{2}{3}C_8 + \frac{r-1}{r}L_3$
	$r \in \{1,2\}$					
28	3	4	4	5	$\mathbb{P}(4,1,1)$	$\frac{2}{3}L_1 + \frac{4}{5}C_4$
29	2	5	5	r	$\mathbb{P}(5,1,1)$	$\frac{1}{2}L_1 + \frac{r-1}{r}C_5$
	$r \in \{7,9\}$					
30	2	3	10	10	$\mathbb{P}(5,1,1)$	$\frac{2}{3}C_{10}$
31	2	3	7	35	$\mathbb{P}(7,1,1)$	$\frac{1}{2}C_7 + \frac{2}{3}L_1 + \frac{4}{5}L_3$
32	2	3	11	11	$\mathbb{P}(11,1,1)$	$\frac{1}{2}C_{11} + \frac{2}{3}L_1$
33	2	3	7	$6r$	$\mathbb{P}(3,2,1)$	$\frac{6}{7}C_6 + \frac{r-1}{r}L_3$
	$r \in \{2,3,4,5,6\}$					
34	2	3	11	12	$\mathbb{P}(3,2,1)$	$\frac{10}{11}C_6 + \frac{1}{2}L_3$
35	2	3	7	$14r$	$\mathbb{P}(7,2,1)$	$\frac{2}{3}C_{14} + \frac{r-1}{r}L_3$
	$r \in \{1,2\}$					
36	2	3	7	21	$\mathbb{P}(7,3,1)$	$\frac{1}{2}C_{21}$

TABLE 2. Case: $\rho(S) > 1$

No.	a_1	a_2	a_3	a_4	$\mathbb{P}(\overline{\mathbf{p}}) \supset S$		$\mathrm{Diff}_S(0)$
37	2	5	5	$2r$	$\mathbb{P}(5,2,2,5)$	$y_1^2+y_2^5+y_3^5+y_4^2$	$\frac{r-1}{r}\Gamma_4$
	$r \in \{3,4\}$						
38	2	4	5	15	$\mathbb{P}(5,5,2,2)$	$y_1^2+y_2^2+y_3^5+y_4^5$	$\frac{1}{2}\Gamma_2 + \frac{2}{3}\Gamma_4$
39	2	3	8	$3r$	$\mathbb{P}(3,2,3,2)$	$y_1^2+y_2^3+y_3^2+y_4^3$	$\frac{3}{4}\Gamma_3 + \frac{r-1}{r}\Gamma_4$
	$r \in \{3,5,7\}$						
40	2	3	9	$2r$	$\mathbb{P}(3,2,2,3)$	$y_1^2+y_2^3+y_3^3+y_4^2$	$\frac{2}{3}\Gamma_3 + \frac{r-1}{r}\Gamma_4$
	$r \in \{5,7,8\}$						
41	3	3	4	10	$\mathbb{P}(2,2,3,3)$	$y_1^3+y_2^3+y_3^2+y_4^2$	$\frac{1}{2}\Gamma_3 + \frac{4}{5}\Gamma_4$
42	2	4	5	10	$\mathbb{P}(5,5,2,1)$	$y_1^2+y_2^2+y_3^5+y_4^{10}$	$\frac{1}{2}\Gamma_2$
43	2	3	8	18	$\mathbb{P}(3,2,3,1)$	$y_1^2+y_2^3+y_3^2+y_4^6$	$\frac{3}{4}\Gamma_3 + \frac{2}{3}\Gamma_4$
44	2	3	10	12	$\mathbb{P}(3,2,3,1)$	$y_1^2+y_2^3+y_3^2+y_4^6$	$\frac{4}{5}\Gamma_3 + \frac{1}{2}\Gamma_4$
45	2	4	6	9	$\mathbb{P}(3,3,1,2)$	$y_1^2+y_2^2+y_3^6+y_4^3$	$\frac{1}{2}\Gamma_2 + \frac{2}{3}\Gamma_4$
46	2	4	6	10	$\mathbb{P}^3$	$y_1^2+y_2^2+y_3^2+y_4^2$	$\frac{1}{2}\Gamma_2 + \frac{2}{3}\Gamma_3 + \frac{4}{5}\Gamma_4$
47	2	3	8	12	$\mathbb{P}(6,4,3,1)$	$y_1^2+y_2^3+y_3^4+y_4^{12}$	$\frac{1}{2}\Gamma_3$
48	3	3	4	6	$\mathbb{P}(2,2,3,1)$	$y_1^3+y_2^3+y_3^2+y_4^6$	$\frac{1}{2}\Gamma_3$
49	2	3	9	12	$\mathbb{P}(3,2,2,1)$	$y_1^2+y_2^3+y_3^3+y_4^6$	$\frac{2}{3}\Gamma_3 + \frac{1}{2}\Gamma_4$
50	2	4	7	7	$\mathbb{P}(7,7,2,2)$	$y_1^2+y_2^2+y_3^7+y_4^7$	$\frac{1}{2}\Gamma_2$
51	3	3	4	8	$\mathbb{P}(4,4,3,3)$	$y_1^3+y_2^3+y_3^4+y_4^4$	$\frac{1}{2}\Gamma_4$
52	3	3	5	5	$\mathbb{P}(5,5,3,3)$	$y_1^3+y_2^3+y_3^5+y_4^5$	0
53	2	4	6	8	$\mathbb{P}(2,1,2,1)$	$y_1^2+y_2^4+y_3^2+y_4^4$	$\frac{2}{3}\Gamma_3 + \frac{1}{2}\Gamma_4$
54	2	4	6	6	$\mathbb{P}(3,3,1,1)$	$y_1^2+y_2^2+y_3^6+y_4^6$	$\frac{1}{2}\Gamma_2$

5.18.3. One can check that all surfaces S in Table 2 have the Picard number $\rho(S) > 1$. Hence, all these singularities are not analytically $\mathbb{Q}$-factorial:

Proposition. *Let (X, P) be an analytic germ of a klt singularity and let $f\colon (Y, S) \to X$ be a plt blow-up such that $f(S) = P$. Then*
(i) $\mathrm{Pic}(Y) = H^2(Y, \mathbb{Z}) = H^2(S, \mathbb{Z}) = \mathrm{Pic}(S)$;
(ii) *if (X, P) is analytically $\mathbb{Q}$-factorial, then* $\mathrm{Pic}(S) = \mathbb{Z}$.

Proof. We have an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_Y \xrightarrow{\exp} \mathcal{O}_Y^* \longrightarrow 0.$$

By Kawamata-Viehweg vanishing $R^i f^* \mathcal{O}_Y = 0$, $i > 0$. Hence, $\mathrm{Pic}(Y) = H^2(Y, \mathbb{Z})$. Similarly, $H^2(S, \mathbb{Z}) = \mathrm{Pic}(S)$. Since Y is an analytic germ near S, $H^2(Y, \mathbb{Z}) = H^2(S, \mathbb{Z})$. If (X, P) is analytically $\mathbb{Q}$-factorial, then $\rho(Y/X) = 1$ and $\mathrm{rk}\,\mathrm{Pic}(S) = 1$. □

Remark. (i) In case $[2, 4, 6, 10]$, $S \simeq \mathbb{P}^1 \times \mathbb{P}^1$. In cases $[3, 3, 4, 6]$ and $[2, 3, 9, 12]$, the projection $S \to \mathbb{P}(2, 2, 1) = \mathbb{P}^2$ is a double cover ramified along $\{(x^3 + y^3 + z^3)z = 0\} \subset \mathbb{P}^2$. Hence, S is a Gorenstein del Pezzo surface of degree 2 having exactly three singular points which are of type A_1. In case $[2, 4, 6, 8]$, $\mathbb{P}(2, 1, 2, 1)$ is isomorphic to a cone over a conic in $\mathbb{P}^4$. Here S is isomorphic to an intersection of two quadrics in $\mathbb{P}^4$. In cases $[2, 3, 8, 18]$, $[2, 3, 10, 12]$ and $[2, 4, 6, 9]$, the surface S has exactly two singular points of type A_2. It is a Gorenstein del Pezzo surface of degree 3, a cubic in $\mathbb{P}^3$.

(ii) In cases [2,3,7,r], $r \in \{11, 13, 17, 19, 23, 25, 29, 31, 37, 41\}$ and [2,3,11,13], the pairs $(S, \varDelta)$ have Shokurov's invariant $\delta(S, \varDelta) = 2$ (see [21, §5], and also [17]). The singularity $[2, 4, 7, 8]$ and all singularities with $S \simeq \mathbb{P}(1, 2, 3)$ give us a log-del Pezzo surface $(S, \varDelta)$ of the so-called "elliptic type" (see [1, Table 1, No. 1]).

References

1. T. Abe, *Classification of exceptional surface complements*, preprint (1998).
2. V. I. Danilov, *Algebraic varieties and schemes*, Encycl. Math. Sci. **23**, Springer, Berlin (1994) 167–297.
3. A. Dimca, *Singularities and coverings of weighted complete intersections*, J. reine u. ange. Math. **366** (1986), 184–193.
4. I. Dolgachev, *Weighted projective varieties*, Lect. Notes Math. **956** (1982) 34–71.
5. W. Fulton, *Introduction to Toric Varieties*, Annals of Math. Studies **131**, Princeton University Press (1993).
6. S. Ishii, *On isolated Gorenstein singularities*, Math. Ann. **270**, (1985) 541–554.
7. S. Ishii, *The canonical modifications by weighted blow-ups*, J. Alg. Geom. **5**, (1996) 783–799.
8. S. Ishii, *The quotients of log-canonical singularities by finite groups*, to appear in Adv. Stud. in Pure Math.
9. S. Ishii, & K. Watanabe, *A geometric characterization of a simple K3-singularity*, Tôhoku Math. J. **44**, (1992) 19–24.

10. Y. Kawamata, *The crepant blowing-up of* 3*-dimensional canonical singularities and its application to the degeneration of surfaces*, Ann. Math. **127**, (1988) 93–163.
11. J. Kollár, *Singularities of pairs*, Proc. Symp. Pure Math. **62**, (1997) 221–287.
12. J. Kollár et al., *Flips and abundance for algebraic threefolds*, A summer seminar at the University of Utah, Salt Lake City, 1991, Astérisque **211**, (1992).
13. D. Markushevich, *Canonical singularities of three dimensional hypersurfaces*, Math. USSR Izv. **26**, (1985) 315–345.
14. D. Markushevich, & Y. Prokhorov, *Klein's group defines an exceptional singularity of dimension* 3, J. Math. Sci. **91**, (1999) no. 1, 1060–1067.
15. D. Markushevich, & Y. Prokhorov, *Exceptional quotient singularities*, to appear in Am. J. Math.
16. Y. Prokhorov, *Blow-ups of canonical singularities*, to appear in Proc. of Conf. in Memory of A.G. Kurosh (1908-1971), Walter der Greither Publ., Berlin.
17. Y. Prokhorov, *Lectures on complements on log surfaces*, preprint.
18. M. Reid, *Canonical threefolds*, Géométrie Algébrique Angers, A. Beauville ed., Sijthoff and Noordhoff (1980) 273–310.
19. K. Saito, *Einfach-ellptische Singularitäten*, Invent. Math. **23**, (1974) 289–325.
20. V.V. Shokurov, 3*-dimensional log flips*, Russian Acad. Sci. Izv. Math. **40**, (1993) 93–202.
21. V.V. Shokurov, *Complements on surfaces*, to appear in J. Math. Sci.
22. A. N. Varchenko, *Zeta-function of monodromy and Newton's diagram*, Invent. Math. **37**, (1976) 253–262.
23. K. Watanabe, & T. Higuchi, *On a certain class of purely elliptic singularities in dimension* > 2, Sci. Rep. Yokohama Nat. Univ. Sect. I, **30**, (1983) 31–35.
24. T. Yonemura, *Hypersurface simple K3-singularities*, Tôhoku Math. J. **42**, (1990) 351–380.

Shihoko Ishii: Department of Mathematics, Tokyo Institute of Technology, Oh-Okayama, Meguro, Tokyo, Japan

Yuri Prokhorov: Department of Mathematics, Tokyo Institute of Technology, Oh-Okayama, Meguro, Tokyo, Japan & Moscow State University, Moscow, Russia